A TENSOR ANALYSIS FOR A V_k IN A PROJECTIVE SPACE S_n*

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1. **Introduction.** In this paper we shall show how an intrinsic tensor analysis may be developed for a curved space or variety V_k of k dimensions immersed in a projective space S_n of $n>k>1$ dimensions. For $n \neq k-1$ there apparently exists no covariant quadratic differential form; so Fubini's method of studying such a variety fails. Or, as Lane suggests,† Fubini's method fails either due to the lack of a quadratic differential form, or to the lack of an absolute calculus for an n-ary p-adic form except when $p=2$.

However, it is well known that an absolute calculus can be developed without the use of a quadratic form by making use of certain generalized Christoffel symbols.‡

These three indexed symbols enable one to introduce into the geometrical theory of a variety the geometry of paths, affine and "projective" connections. In that manner certain tensors and vectors arising in those theories can be expressed in terms of tensors and vectors arising in the study of the variety from the point of view of classical projective geometry. In particular, the Weyl projective curvature tensor is expressible in terms of tensors arising in the classical geometric theory of a variety V_k.

Finally, we show that a generalized Riemann space of k dimensions with a fundamental symmetric connection characterizing the space may be considered as being immersed in a projective space of $n=k(k+3)/2$ dimensions. This theorem is an evident generalization of the fact that a Riemann space may always be considered as immersed in an euclidean space of sufficiently high dimension.

2. **The fundamental differential equations.** Let the homogeneous projective coordinates x^i, $(i=1, 2, \cdots, n+1)$, of a point P in S_n be given as analytic functions of exactly k parameters $u^1, u^2, \cdots, u^k$:

$$x^i = x^i(u^1, u^2, \cdots, u^k). \tag{1}$$

The totality of such points P we shall call *a variety* V_k.

The functions x and $\partial x/\partial u^\rho$ may be interpreted as the homogeneous projective coordinates of $k+1$ points. These points determine a cer-

* An address delivered before the Cleveland meeting of the Society on November 25, 1938, by invitation of the Program Committee.

† Lane [1, p. 292].

‡ See, for example, Eisenhart [2].

tain linear space T_k of k dimensions which we shall call the *tangent space of V_k at x.*

Let ${}_{(s)}y^i$, $(s=1, 2, \cdots, r=n-k)$, be $r(n+1)$ other functions of $u^1, u^2, \cdots, u^k$ such that the determinant

$$R = \left(x, \frac{\partial x}{\partial u^1}, \frac{\partial x}{\partial u^2}, \cdots, \frac{\partial x}{\partial u^k}, {}_{(1)}y, {}_{(2)}y, \cdots, {}_{(r)}y\right) \tag{2}$$

does not vanish. Then the $n+1$ functions x and the $r(n+1)$ functions ${}_{(s)}y$ satisfy a system of differential equations of the form

$$\begin{aligned} \frac{\partial^2 x}{\partial u^\alpha \partial u^\beta} &= L^\rho_{\alpha\beta} \frac{\partial x}{\partial u^\rho} + p_{\alpha\beta} x + {}_{(s)}D_{\alpha\beta}\, {}_{(s)}y, \\ \frac{\partial\, {}_{(s)}y}{\partial u^\alpha} &= {}_{(s)}M^\rho_\alpha \frac{\partial x}{\partial u^\rho} + {}_{(s)}q_\alpha x + {}_{(st)}E_\alpha\, {}_{(t)}y. \end{aligned} \tag{3}$$

We shall use the usual umbral convention: any index repeated in any symbol or group of symbols denotes summation over that index. The Greek letters $\alpha, \beta, \gamma, \cdots, \rho, \sigma$ shall be understood to have the range from 1 to k, the letters i, j from 1 to $n+1$, and the letters s, p, l, m from 1 to r.

From (3) we note that $L^\rho_{\alpha\beta} = L^\rho_{\beta\alpha}$, $p_{\alpha\beta} = p_{\beta\alpha}$, ${}_{(s)}D_{\alpha\beta} = {}_{(s)}D_{\beta\alpha}$. Moreover any other solution X^i, ${}_{(s)}Y^i$ of (3) is of the form

$$X^i = a^i_j x^j, \qquad {}_{(s)}Y^i = a^i_j\, {}_{(s)}y^j, \qquad a = |a^i_j| \neq 0. \tag{4}$$

Hence the coefficients of system (3) are invariant under the projective transformation (4).

The coefficients of system (3) are not independent. They satisfy certain integrability conditions. These conditions are

$$\begin{aligned} P^\rho_{\alpha\beta\gamma} &= R^\rho_{\alpha\beta\gamma} + p_{\alpha\gamma}\delta^\rho_\beta - p_{\alpha\beta}\delta^\rho_\gamma \\ &= {}_{(s)}D_{\alpha\beta}\, {}_{(s)}M^\rho_\gamma - {}_{(s)}D_{\alpha\gamma}\, {}_{(s)}M^\rho_\beta, \\ p_{\alpha\beta,\gamma} - p_{\alpha\gamma,\beta} &= {}_{(s)}q_\beta\, {}_{(s)}D_{\alpha\gamma} - {}_{(s)}q_\gamma\, {}_{(s)}D_{\alpha\beta}, \\ {}_{(s)}D_{\alpha\beta,\gamma} - {}_{(s)}D_{\alpha\gamma,\beta} &= {}_{(ts)}E_\beta\, {}_{(t)}D_{\alpha\gamma} - {}_{(ts)}E_\gamma\, {}_{(t)}D_{\alpha\beta}, \\ {}_{(s)}M^\rho_{\alpha,\beta} - {}_{(s)}M^\rho_{\beta,\alpha} &= \delta^\rho_\alpha\, {}_{(s)}q_\beta - \delta^\rho_\beta\, {}_{(s)}q_\alpha \\ &\quad + {}_{(st)}E_\beta\, {}_{(t)}M^\rho_\alpha - {}_{(st)}E_\alpha\, {}_{(t)}M^\rho_\beta, \\ {}_{(s)}q_{\alpha,\beta} - {}_{(s)}q_{\beta,\alpha} &= {}_{(st)}E_\beta\, {}_{(t)}q_\alpha - {}_{(st)}E_\alpha\, {}_{(t)}q_\beta \\ &\quad + p_{\alpha\sigma}\, {}_{(s)}M^\sigma_\beta - p_{\beta\sigma}\, {}_{(s)}M^\sigma_\alpha, \\ {}_{(st)}E_{\alpha,\beta} - {}_{(st)}E_{\beta,\alpha} &= {}_{(sp)}E_\beta\, {}_{(pt)}E_\alpha - {}_{(sp)}E_\alpha\, {}_{(pt)}E_\beta \\ &\quad + {}_{(t)}D_{\alpha\rho}\, {}_{(s)}M^\rho_\beta - {}_{(t)}D_{\rho\beta}\, {}_{(s)}M^\rho_\alpha, \end{aligned} \tag{5}$$

wherein

$$R^{\rho}_{\alpha\beta\gamma} = \frac{\partial}{\partial u^{\beta}} L^{\rho}_{\alpha\gamma} - \frac{\partial}{\partial u^{\gamma}} L^{\rho}_{\alpha\beta} + L^{\sigma}_{\alpha\gamma} L^{\rho}_{\sigma\beta} - L^{\sigma}_{\alpha\beta} L^{\rho}_{\sigma\gamma}. \tag{6}$$

The other abbreviations occurring in (5) will be explained at a later time.

By direct calculation from the integrability conditions or by proper changes in notation in a paper by Lane,* we find that

$$\frac{\partial}{\partial u^{\alpha}} [L^{\rho}_{\sigma\beta} + {}_{(ss)}E_{\beta}] = \frac{\partial}{\partial u^{\beta}} [L^{\rho}_{\sigma\alpha} + {}_{(ss)}E_{\alpha}]. \tag{7}$$

Hence there exists a function f such that

$$f_{\sigma} = \frac{\partial f}{\partial u^{\sigma}} = L^{\rho}_{\sigma\rho} + {}_{(ss)}E_{\sigma}. \tag{8}$$

3. **Tensors on V_k.** The variety V_k is not changed if on the differential equations (3) we make the transformations

$$u^{\alpha} = u^{\alpha}(\bar{u}^1, \bar{u}^2, \cdots, \bar{u}^k), \qquad \Delta = \left| \frac{\partial u^{\alpha}}{\partial \bar{u}^{\beta}} \right| \neq 0; \tag{9}$$

$$x = \lambda \bar{x}, \qquad \lambda \neq 0; \tag{10}$$

$${}_{(s)}y = {}_{(s)}\theta^{\rho} \frac{\partial x}{\partial u^{\rho}} + {}_{(s)}\phi x + a_{st}\, {}_{(t)}\bar{y}, \qquad A = |a_{st}| \neq 0. \tag{11}$$

Under (9) the differential equations (3) assume a form wherein the new coefficients $\bar{L}^{\rho}_{\alpha\beta}$ are given by

$$\bar{L}^{\rho}_{\alpha\beta} = \left[L^{\lambda}_{\gamma\delta} \frac{\partial u^{\gamma}}{\partial \bar{u}^{\alpha}} \frac{\partial u^{\delta}}{\partial \bar{u}^{\beta}} + \frac{\partial^2 u^{\lambda}}{\partial \bar{u}^{\alpha} \partial \bar{u}^{\beta}} \right] \frac{\partial \bar{u}^{\rho}}{\partial u^{\lambda}}. \tag{12}$$

Moreover the coefficients $p_{\alpha\beta}$, ${}_{(s)}D_{\alpha\beta}$ transform by (9) like the covariant components of tensors of the second order; and ${}_{(s)}q_{\alpha}$, ${}_{(st)}E_{\alpha}$ transform like the covariant components of vectors (or tensors of the first order); and ${}_{(s)}M^{\rho}_{\alpha}$ transforms like the components of a mixed tensor of the second order. Accordingly, we call them the contravariant and covariant components of tensors of the kind and order indicated by their indices.

We may readily verify that $P^{\rho}_{\alpha\beta\gamma}$, $R^{\rho}_{\alpha\beta\gamma}$ are components of tensors of the fourth order, contravariant of the first, and covariant of the

* Lane [3, p. 796].

third. We shall call the tensor $P^{\rho}_{\alpha\beta\gamma}$ *the projective curvature tensor of* V_k *relative to the space* N_r of points x, $_{(\sigma)}y$.

Unfortunately these vectors and tensors have no geometric significance since, although invariant under the projective transformation (4), they are not invariant under (10) and (11). To this end we normalize the coordinates x themselves, and the space of points N_r determined by x and $_{(\sigma)}y$.

4. **A semi-canonical form of the differential equations.** Differential geometers have used various methods for reducing the differential equations, forms, or power series arising in their respective theories to canonical forms. In the case $k=2$, $r=1$ (that is, in the case of an ordinary surface in three dimensional projective space S_3), Fubini normalized the coordinates x by a special transformation (10) which made the ratio of the discriminants of certain covariant differential forms (one quadratic and one cubic) a constant.* In the case $k=n-1$, $r=1$ (that is, in the case of a hypersurface in S_n), Hlavatý used one covariant quadratic form, covariant differentiation with respect to the form, and the properties of affine connection to derive a canonical form of his differential equations.†

Wilczynski, starting with a defining set of differential equations, observed the effect of transformations of the arbitrary parameters in the differential equations which left invariant the configuration he was studying, and then by a judicious particular choice of transformation reduced his set to a canonical form.‡ He also always computed a complete system of invariants and covariants for his configuration. A variation of his method for $k=2$, $r=1$ consists in starting with a Taylor's expansion of one nonhomogeneous projective coordinate in terms of the other two, and then, by making use of available transformations of the arbitrary parameters, reducing the power series to a simple or convenient canonical form. A weather eye was always held out, however, for a form to which a more or less simple geometrical significance could be attached. This method§ has been used extensively by Lane, Stouffer, Green, and others.

We use an adaptation‖ of the method used by Grove for reducing our defining differential equations to a canonical form. We observe first that the functions f_σ defined by (8) transform by (9), (10), (11) into $\bar{f}_\sigma$ by the formula

* Fubini and Čech [4, 5].
† Hlavatý [6].
‡ Wilczynski [7, 8, 9].
§ Lane [10].
‖ Grove [11].

$$\bar{f}_{\bar{\sigma}} = \frac{\partial u^{\rho}}{\partial \bar{u}^{\sigma}} \left[f_{\rho} + \frac{\partial}{\partial u^{\rho}} \log \frac{\Delta}{A\lambda^{k+1}} \right]. \tag{13}$$

Let R be a function of $u^1, u^2, \cdots, u^k$ with the following properties: (a) $R \neq 0$, (b) the transform $\bar{R}$ of R by (9), (10), (11) is given by

$$\bar{R} = \lambda^k A R/\Delta. \tag{14}$$

Now define the function p_{σ} by the expression

$$p_{\sigma} = f_{\sigma} + \frac{\partial}{\partial u^{\sigma}} \log R. \tag{15}$$

The point whose homogeneous projective coordinates are defined by the formula

$$r_{\sigma} = \frac{\partial x}{\partial u^{\sigma}} + p_{\sigma} x \tag{16}$$

transforms by (9), (10), (11) into $\bar{r}_{\bar{\sigma}}$ where

$$\bar{r}_{\bar{\sigma}} = \lambda \frac{\partial u^{\rho}}{\partial \bar{u}^{\sigma}} \left[\frac{\partial \bar{x}}{\partial u^{\rho}} + \left(p_{\rho} - \frac{\partial}{\partial u^{\rho}} \log \lambda \right) \bar{x} \right]. \tag{17}$$

Hence if we choose λ so that

$$\frac{\partial}{\partial u^{\rho}} \log \lambda = p_{\rho},$$

we make

$$r_{\sigma} = \frac{\partial x}{\partial u^{\sigma}}, \tag{18}$$

and from (17)

$$\bar{r}_{\bar{\sigma}} = \lambda \frac{\partial u^{\rho}}{\partial \bar{u}^{\sigma}} r_{\rho}.$$

Hence r_{σ} is an intrinsic covariant vector.

The form of (18) of course depends upon the choice of the function R. In particular, the function

$$R = \left(x, \frac{\partial x}{\partial u^1}, \frac{\partial x}{\partial u^2}, \cdots, \frac{\partial x}{\partial u^k}, {}_{(1)}y, {}_{(2)}y, \cdots, {}_{(r)}y \right)$$

has the desired properties (a) and (b), and could be used in deriving (18). The function R may be chosen conveniently to the problem at

hand. The coordinates x so normalized cause the points whose coordinates are $\partial x/\partial w^\rho$ *to lie in a covariant flat space* R_{k-1} of $k-1$ *dimensions lying in the tangent space* T_k.

Now if in the transformation (11) we choose ${}_{(s)}\theta^\rho$ and ${}_{(s)}\phi$ to satisfy the differential equations

$$
\begin{aligned}
\frac{\partial\, {}_{(s)}\theta^\sigma}{\partial u^\alpha} &= {}_{(s)}M^\sigma_\alpha - {}_{(s)}\theta^\rho L^\sigma_{\alpha\rho} - {}_{(s)}\phi\delta^\sigma_\alpha + ({}_{(st)}E_\alpha - {}_{(s)}\theta^\rho\ {}_{(t)}D_{\alpha\rho})_{(t)}\theta^\sigma, \\
\frac{\partial\, {}_{(s)}\phi}{\partial u^\alpha} &= {}_{(s)}q_\alpha - {}_{(s)}\theta^\rho p_{\alpha\rho} + ({}_{(st)}E_\alpha - {}_{(s)}\theta^\rho\ {}_{(t)}D_{\alpha\rho})_{(t)}\phi,
\end{aligned}
\tag{19}
$$

whose integrability conditions appear among those of system (3), we cause (3) to assume a form for which

$$
{}_{(s)}M^\rho_\alpha = {}_{(s)}q_\alpha = 0, \qquad s = 1, 2, \cdots, r;\ \alpha, \rho = 1, 2, \cdots, k.
$$

Moreover, we may choose A of (11) so that ${}_{(ss)}E_\alpha = 0$. Hence by proper choice of the space N_r of points x, ${}_{(s)}y$ we may cause the system (3) to assume *a semi-canonical form*

$$
\begin{aligned}
\frac{\partial^2 x}{\partial u^\alpha \partial u^\beta} &= L^\rho_{\alpha\beta}\frac{\partial x}{\partial u^\rho} + p_{\alpha\beta}x + {}_{(s)}D_{\alpha\beta}\ {}_{(s)}y, \\
\frac{\partial\, {}_{(s)}y}{\partial u^\alpha} &= {}_{(st)}E_\alpha\ {}_{(t)}y,
\end{aligned}
\tag{20}
$$

characterized by

$$
L^\rho_{\rho\alpha} + \frac{\partial}{\partial u^\alpha}\log R = {}_{(ss)}E_\alpha = 0.
$$

The space N_r of r dimensions determined by x and ${}_{(s)}y$ giving rise to the form (20) may be called a *projective normal space*. Its precise nature depends upon the particular choice of the function R.

The form of (20) is preserved under all transformations of the form

$$
\begin{aligned}
u^\alpha &= u^\alpha(\bar{u}^1, \bar{u}^2, \cdots, \bar{u}^k), \qquad x = c\bar{x}, \quad c \text{ a constant}, \\
{}_{(s)}y &= a_{st}\ {}_{(t)}\bar{y}, \qquad\qquad A \text{ a constant}.
\end{aligned}
$$

5. **Covariant differentiation.** The tensor $p_{\alpha\beta}$ of the semi-canonical form could be used as a basis for an absolute calculus. We notice, however, that the functions $L^\rho_{\alpha\beta}$ transform under (9) by the same law of transformation as that by which the Christoffel symbols transform. Hence these functions can be used as a basis for covariant differentiation.

As defined* by Veblen and Thomas, "covariant differentiation is a process by which from a given tensor, there may be found a new tensor with one more covariant index." The definitions we use are the usual ones, and bear a close analogy to covariant differentiation in euclidean space S_3.

We denote covariant differentiation with respect to $L^{\rho}_{\alpha\beta}$ by a comma. In particular,

$$x_{,\alpha} = \frac{\partial x}{\partial u^{\alpha}}; \tag{21}$$

$$x_{,\alpha\beta} = \frac{\partial^2 x}{\partial u^{\alpha}\partial u^{\beta}} - L^{\rho}_{\alpha\beta}\frac{\partial x}{\partial u^{\rho}}. \tag{22}$$

But from (20), $x_{,\alpha\beta}$ may be written

$$x_{,\alpha\beta} = p_{\alpha\beta}x + {}_{(s)}D_{\alpha\beta}\,{}_{(s)}y. \tag{23}$$

Hence, *similar to euclidean geometry of n dimensions, the second covariant derivative of x gives a point (or a line through x) in the unique normal space.*

Now if one differentiates the equations (3) covariantly with respect to $L^{\rho}_{\alpha\beta}$, one finds the integrability conditions in covariant derivative form. These conditions are given by (5) wherein the comma is to be interpreted as the symbol of covariant differentiation. Using system (20) instead of (3), we may write the integrability conditions in the simple form

$$\begin{gathered} P^{\rho}_{\alpha\beta\gamma} = 0, \qquad p_{\alpha\beta,\gamma} - p_{\alpha\gamma,\beta} = 0, \\ {}_{(s)}D_{\alpha\beta,\gamma} - {}_{(s)}D_{\alpha\gamma,\beta} = {}_{(ts)}E_{\beta}\,{}_{(t)}D_{\alpha\gamma} - {}_{(ts)}E_{\gamma}\,{}_{(t)}D_{\alpha\beta}, \\ {}_{(st)}E_{\alpha,\beta} - {}_{(st)}E_{\beta,\alpha} = {}_{(sp)}E_{\beta}\,{}_{(pt)}E_{\alpha} - {}_{(sp)}E_{\alpha}\,{}_{(pt)}E_{\beta}. \end{gathered} \tag{24}$$

6. **Affine and "projective" connections.** The functions $L^{\rho}_{\alpha\beta}$ may be used to define an affine connection† on V_k since by (12) they transform according to the law of such a connection.‡ Since the functions $L^{\rho}_{\alpha\beta}$ are invariant under the projective transformation (4) in S_n, whatever may be valid for a given V_k in S_n is equally valid for any projective transform of V_k in S_n. We may therefore instigate a study of the geometry of paths on V_k from the point of view of classical differential geometry by the use of this connection.

* Veblen and Thomas [12, p. 569].

† Such a connection is said to be of zero torsion, since $L^{\rho}_{\alpha\beta} = L^{\rho}_{\beta\alpha}$. The term "torsion" was introduced by Eddington [13].

‡ Schouten [14].

A curve on V_k will be said to be a path if the functions $u^\alpha = u^\alpha(t)$ defining the curve satisfy the differential equations

$$\frac{d^2u^\rho}{dt^2} + L^\rho_{\alpha\beta}\frac{du^\alpha}{dt}\frac{du^\beta}{dt} = 0. \tag{25}$$

It may readily be proved that the osculating plane of the curve $u^\alpha = u^\alpha(v)$ at x intersects the space N_r in a line if and only if the functions $u^\alpha(v)$ satisfy equations of the form

$$\frac{d^2u^\rho}{dv^2} + L^\rho_{\alpha\beta}\frac{du^\alpha}{dv}\frac{du^\beta}{dv} = \theta\frac{du^\rho}{dv}, \qquad \theta \text{ arbitrary}. \tag{26}$$

But by proper choice of the function $t = t(v)$, we may write (26) in the form (25). Curves whose osculating planes intersect a given space N_r in lines are said to form an axial system.* In a sense, therefore, *the paths determined by the connection $L^\rho_{\alpha\beta}$ play the role of geodesics on a variety V_k in euclidean space S_n since the osculating planes of such geodesics intersect the unique normal space in lines.*

Consider now the so-called "projective" connection†

$$\Pi^\rho_{\alpha\beta} = L^\rho_{\alpha\beta} - \frac{\delta^\rho_\alpha L^\sigma_{\sigma\beta}}{k+1} - \frac{\delta^\rho_\beta L^\sigma_{\sigma\alpha}}{k+1}. \tag{27}$$

In view of the conditions imposed on the differential equations (3) to reduce them to the form (20), we may write (27) in the form

$$\Pi^\rho_{\alpha\beta} = L^\rho_{\alpha\beta} + \delta^\rho_\alpha\frac{\partial}{\partial u^\beta}\log R' + \delta^\rho_\beta\frac{\partial}{\partial u^\alpha}\log R', \tag{28}$$

wherein

$$R' = R^{-1/(k-1)}.$$

Hence $\Pi^\rho_{\alpha\beta}$ *defines the same set of paths*‡ *as* $L^\rho_{\alpha\beta}$. Therefore $\Pi^\rho_{\alpha\beta}$ *defines a projective connection in the sense of Thomas and, moreover, is projective in the sense of classical projective geometry.*

If we let $\mathfrak{B}^\rho_{\alpha\beta\gamma}$ be the curvature tensor for the connection $\Pi^\rho_{\alpha\beta}$, and let

$$r_{\alpha\beta} = \frac{\mathfrak{B}^\rho_{\alpha\beta\rho}}{k-1},$$

the Weyl projective curvature tensor $W^\rho_{\alpha\beta\gamma}$ is defined by

* Bortolotti [15].

† Thomas [16].

‡ Thomas [16] and Bortolotti [17].

$$W^{\rho}_{\alpha\beta\gamma} = \mathfrak{B}^{\rho}_{\alpha\beta\gamma} + \delta^{\rho}_{\beta}\Gamma_{\alpha\gamma} - \delta^{\rho}_{\gamma}\Gamma_{\alpha\beta}. \tag{29}$$

We find that

$$W^{\rho}_{\alpha\beta\gamma} = R^{\rho}_{\alpha\beta\gamma} + \frac{\delta^{\rho}_{\beta}R^{\sigma}_{\alpha\gamma\sigma}}{k-1} - \frac{\delta^{\rho}_{\gamma}R^{\sigma}_{\alpha\beta\sigma}}{k-1} = P^{\rho}_{\alpha\beta\gamma} + \frac{\delta^{\rho}_{\beta}P^{\sigma}_{\alpha\gamma\sigma}}{k-1} - \frac{\delta^{\rho}_{\gamma}P^{\sigma}_{\alpha\beta\sigma}}{k-1}. \tag{30}$$

Hence

$$W^{\rho}_{\alpha\beta\gamma} = 0.$$

It follows, therefore, that the choice of the space N_r giving rise to the semi-canonical form (20) implies that the variety V_k, $(k>2)$, is projectively plane.* Moreover, for $k>2$ there is a preferred choice of parameters u^{α} such that $L^{\rho}_{\alpha\beta}$ are zero. Hence with $k>2$, and by proper choice of the transformation (10) we may cause system (20) to assume the form

$$\frac{\partial^2 x}{\partial u^{\alpha}\partial u^{\beta}} = p_{\alpha\beta}x + {}_{(s)}D_{\alpha\beta}\,{}_{(s)}y, \qquad \frac{\partial\,{}_{(s)}y}{\partial u^{\alpha}} = {}_{(st)}E_{\alpha}\,{}_{(t)}y. \tag{31}$$

7. **Other connections on V_k.** By proper, but not unique, choice of the functions ${}_{(s)}\theta^{\rho}$, ${}_{(s)}\phi$ in transformation (11), and with certain limitations on the tensors ${}_{(s)}D_{\alpha\beta}$, we may make

$${}_{(ss)}E_{\alpha} = {}_{(t)}M^{\sigma}_{\sigma} = 0, \qquad \alpha = 1, 2, 3, \cdots, k; t = 1, 2, \cdots, r. \tag{32}$$

Under these conditions, the Weyl "projective" curvature tensor is still given by (30). Under the conditions (32) we may write (30) in the form

$$W^{\rho}_{\alpha\beta\gamma} = {}_{(s)}D_{\alpha\sigma}\,{}_{(s)}Q^{\rho\sigma}_{\beta\gamma} \tag{33}$$

wherein

$${}_{(s)}Q^{\rho\sigma}_{\beta\gamma} = \delta^{\sigma}_{\beta}\,{}_{(s)}M^{\rho}_{\gamma} - \delta^{\sigma}_{\gamma}\,{}_{(s)}M^{\rho}_{\beta} + \frac{\delta^{\rho}_{\gamma}\,{}_{(s)}M^{\sigma}_{\beta}}{k-1} - \frac{\delta^{\rho}_{\beta}\,{}_{(s)}M^{\sigma}_{\gamma}}{k-1}.$$

The Weyl projective curvature tensor vanishes for $k=2$. Moreover, we may show that the geometry of paths under the conditions (32) possesses an invariant† integral of the form

$$\int R^{-1}du^1du^2\cdots du^k. \tag{34}$$

Similarly, the geometry of paths, based on the conditions giving rise

* Bortolotti [17].

† Veblen [18]; Eisenhart [19].

to the semi-canonical form (20), also possesses an invariant integral.

For a general connection $L^{\rho}_{\alpha\beta}$ based on the defining system (3), the projective connection $\Pi^{\rho}_{\alpha\beta}$ transforms by (11) into $\overline{\Pi}^{\rho}_{\alpha\beta}$ by the formula

$$\overline{\Pi}^{\rho}_{\alpha\beta} = \Pi^{\rho}_{\alpha\beta} + {}_{(s)}\theta^{\sigma}\ {}_{(s)}Q^{\rho}_{\alpha\beta\sigma}, \tag{35}$$

wherein

$${}_{(s)}Q^{\rho}_{\alpha\beta\sigma} = \delta^{\rho}_{\sigma}\ {}_{(s)}D_{\alpha\beta} - \frac{\delta^{\rho}_{\alpha}\ {}_{(s)}D_{\sigma\beta}}{k+1} - \frac{\delta^{\rho}_{\beta}\ {}_{(s)}D_{\alpha\sigma}}{k+1}. \tag{36}$$

Hence the geometry of paths based on the projective connection $\Pi^{\rho}_{\alpha\beta}$ is independent of the space N_r if and only if the tensors ${}_{(s)}Q^{\rho}_{\alpha\beta\sigma}$ vanish; that is, if and only if the variety V_k is a linear space.

8. **The parallel displacement of Levi-Civita.** Consider given the contravariant components of a vector λ^{ρ} and a curve C with parametric equations $u^{\alpha}=u^{\alpha}(t)$ on V_k. If the components λ^{ρ} satisfy the differential equations

$$\frac{d\lambda^{\rho}}{dt} + L^{\rho}_{\alpha\beta}\lambda^{\alpha}\frac{du^{\beta}}{dt} = 0, \tag{37}$$

it is said that *the vector has suffered a parallel displacement along C.* If (37) is satisfied for all curves C through x on V_k, then

$$\lambda^{\rho}_{,\alpha} = \frac{\partial\lambda^{\rho}}{\partial u^{\alpha}} + L^{\rho}_{\alpha\sigma}\lambda^{\sigma} = 0, \tag{38}$$

and *it is said of the field of contravariant vectors that they are parallel with respect to any curve.*

Now consider any point z whose coordinates are determined by the expression

$$vz = \mu x + \lambda^{\rho}\frac{\partial x}{\partial u^{\rho}} + a_s\ {}_{(s)}y. \tag{39}$$

As x moves along a curve C_x ($u^{\alpha}=u^{\alpha}(t)$) on V_k, z moves along a curve C_z on a variety V'_k. The tangent to C_z at z is determined by z and by the point defined by the expression

$$\begin{aligned}\frac{d(vz)}{dt} = \frac{d(\mu x)}{dt} + \frac{\partial x}{\partial u^{\rho}}\left[\frac{d\lambda^{\rho}}{dt} + L^{\rho}_{\alpha\beta}\lambda^{\alpha}\frac{du^{\beta}}{dt}\right] \\ + \left[p_{\rho\beta}\lambda^{\rho}x + \left(\frac{\partial a_s}{\partial u^{\beta}} + {}_{(ts)}E_{\beta a_t} + \lambda^{\alpha}\ {}_{(s)}D_{\alpha\beta}\right){}_{(s)}y\right]\frac{du^{\beta}}{dt}.\end{aligned} \tag{40}$$

Hence the tangent to C_s at z and the tangent to C_z at x are coplanar if and only if

$$\frac{d\lambda^\rho}{dt} + L^\rho_{\alpha\beta}\lambda^\alpha \frac{du^\beta}{dt} = 0,$$

$$\frac{da_s}{dt} + ({}_{(ts)}E_\beta a_t + \lambda^\alpha {}_{(s)}D_{\alpha\beta}) \frac{du^\beta}{dt} = 0. \tag{41}$$

If equations (41) hold, we may say that C_z and C_s are in relation C, *a sort of generalization of the transformation of Combescure.* If (41) is to hold for all curves on V_k, then λ^ρ and a_s must satisfy the equations

$$\lambda^\rho_{,\alpha} = \frac{\partial\lambda^\rho}{\partial u^\alpha} + L^\rho_{\sigma\alpha}\lambda^\sigma = 0,$$

$$\frac{\partial a_s}{\partial u^\alpha} + {}_{(ts)}E_\alpha a_t + \lambda^\sigma {}_{(s)}D_{\alpha\sigma} = 0. \tag{42}$$

The tangents to any curve on V_k intersect the tangents to the corresponding curves on V_k'. We may say that V_k and V_k' are *in relation C.* Granted that the first of equations* (42) has solutions, the integrability conditions of the last of (42) follow from those of system (3).

The first of (42) has the "trivial" solution $\lambda^\rho = 0$. Then the tangents to the corresponding curves on V_k and V_k' are coplanar if and only if a_s satisfies the equations

$$\frac{\partial a_s}{\partial u^\alpha} + {}_{(ts)}E_\alpha a_t = 0. \tag{43}$$

The integrability conditions of (43) follow from those of (3). But *if* (43) *is satisfied, the line xz passes through a fixed point.* We shall say that V_k and V_k' are *in the relation of a radial transformation.* If $k=2$, $r=1$, and the varieties V_k and V_k' are ordinary surfaces in S_3, then *the two surfaces are in the relation of a radial transformation* as the term is usually understood.

9. **Conclusion.** Finally, in conclusion, let there be given a generalized Riemann space† S_k the coordinates of whose points are $u^1, u^2, \cdots, u^k$. Let there be given a fundamental affine connection $\Lambda^\rho_{\alpha\beta}$ determining the structure of the space. Suppose, moreover, that $\Lambda^\rho_{\alpha\beta}$ is a symmetric connection and that under the transformation

$$u^\alpha = u^\alpha(\bar{u}^1, \bar{u}^2, \cdots, \bar{u}^k)$$

* Meyer and Thomas [20].

† Thomas [21]; Hlavatý [22].

the functions $\Lambda^{\rho}_{\alpha\beta}$ suffer the usual transformation

$$\bar{\Lambda}^{\rho}_{\alpha\beta} = \left[\Lambda^{\lambda}_{\gamma\delta}\frac{\partial u^{\gamma}}{\partial \bar{u}^{\alpha}}\frac{\partial u^{\delta}}{\partial \bar{u}^{\beta}} + \frac{\partial^2 u^{\lambda}}{\partial \bar{u}^{\alpha}\partial \bar{u}^{\beta}}\right]\frac{\partial \bar{u}^{\rho}}{\partial u^{\lambda}} \tag{44}$$

of such connections. The structure of the space S_k is equally as well determined by a system of paths on S_k: $u^{\alpha} = u^{\alpha}(t)$, where the functions $u^{\alpha}(t)$ satisfy the usual differential equations of such paths, namely

$$\frac{d^2u^{\rho}}{dt^2} + \Lambda^{\rho}_{\alpha\beta}\frac{du^{\alpha}}{dt}\frac{du^{\beta}}{dt} = 0. \tag{45}$$

Let the space S_k be mapped on a variety V_k immersed in a projective space S_n of $n = k(k+3)/2$ dimensions. Then the $n+1$ homogeneous projective coordinates of a point P on V_k are expressible as functions of $u^1, u^2, \cdots, u^k$. Let ${}_{(s)}y$ be the homogeneous projective coordinates of $r = n(n+1)/2$ other points in S_n such that

$$R = \left(x, \frac{\partial x}{\partial u^1}, \frac{\partial x}{\partial u^2}, \cdots, \frac{\partial x}{\partial u^k}, {}_{(1)}y, {}_{(2)}y, \cdots, {}_{(r)}y\right) \neq 0.$$

Then the functions x and ${}_{(s)}y$ satisfy a system of equations of the form (3).

By a transformation of the form (11) we may make

$$L^{\rho}_{\alpha\beta} + {}_{(s)}\theta^{\rho}\,{}_{(s)}D_{\alpha\beta} = \Lambda^{\rho}_{\alpha\beta}, \tag{46}$$

since the rank of the matrix $({}_{(s)}D_{\alpha\beta})$ is r. Hence we may cause equations (3) to assume the form

$$\begin{aligned}\frac{\partial^2 x}{\partial u^{\alpha}\partial u^{\beta}} &= \Lambda^{\rho}_{\alpha\beta}\frac{\partial x}{\partial u^{\rho}} + p_{\alpha\beta}x + {}_{(s)}D_{\alpha\beta}\,{}_{(s)}y,\\ \frac{\partial\,{}_{(s)}y}{\partial u^{\alpha}} &= {}_{(s)}M^{\rho}_{\alpha}\frac{\partial x}{\partial u^{\rho}} + {}_{(s)}q_{\alpha}x + {}_{(st)}E_{\alpha}\,{}_{(t)}y.\end{aligned} \tag{47}$$

With Thomas we may describe the "*projective*" *theory of connections as the study of all possible affine connections.* From this point of view *the study of equations* (46) *for all possible choices of the space* N_r *of points* x, ${}_{(s)}y$ (that is, of all possible choices of ${}_{(s)}\theta^{\rho}$) *is the projective study of connections in the symmetric case.*

Moreover, under the transformation (10) the coefficients $\bar{\Lambda}^{\rho}_{\alpha\beta}$ transform by (10) according to the formula

$$\bar{\Lambda}^{\rho}_{\alpha\beta} = \Lambda^{\rho}_{\alpha\beta} - \delta^{\rho}_{\alpha}\frac{\partial}{\partial u^{\beta}}\log\lambda - \delta^{\rho}_{\beta}\frac{\partial}{\partial u^{\alpha}}\log\lambda.$$

Hence* $\overline{\Lambda}^{\rho}_{\alpha\beta}$ *defines on* S_k *the same system of paths as do* $\Lambda^{\rho}_{\alpha\beta}$. We may state our result in the following form:

Let the space S_k *of structure* $\Lambda^{\rho}_{\alpha\beta}$ *be mapped on a variety* V_k *in a projective space* S_n *of* $n=k(k+3)/2$ *dimensions. Then the paths of the space* S_k *map into the curves of the variety* V_k *whose osculating planes intersect an unique space of points* N_r *of* $r=k(k+1)/2$ *dimensions in lines.* In other words, *a generalized Riemann space with a given structure may be considered as immersed in a projective space of sufficiently high dimension.*

In particular, if we first normalize the coordinates x by a transformation (10), and then choose the space N_r of points x, ${}_{(\sigma)}y$, we may make the connection $L^{\rho}_{\alpha\beta}$ of (3) assume the form $\{\rho, \alpha\beta\}$ of the Christoffel symbols obtained from a so-called fundamental metric tensor $g_{\alpha\beta}$ of a Riemann space S_k. *The space* N_r *would then be uniquely determined by the metric tensor, and any projective transform of* V_k *in* S_n *would possess the same Riemannian metric.* The geodesics from this Riemann geometric point of view would be the curves on V_k whose osculating planes intersect an unique normal space N_r in lines just as occurs in the euclidean geometry of a variety V_k in S_n.

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MICHIGAN STATE COLLEGE

THE STIELTJES MOMENT PROBLEM FOR FUNCTIONS OF BOUNDED VARIATION

R. P. BOAS, JR.*

1. **Introduction.** We shall establish the following theorem, which at first sight appears quite unexpected:

THEOREM 1. *Any sequence* $\{\mu_n\}$ *of real numbers can be represented in the form*

$$\mu_n = \int_0^\infty t^n d\alpha(t), \qquad n = 0, 1, 2, \cdots, \tag{1.1}$$
$$\int_0^\infty |d\alpha(t)| < \infty.$$

The problem of determining necessary and sufficient conditions for a sequence of numbers $\{\mu_n\}$ to have the form

$$\mu_n = \int_0^\infty t^n d\alpha(t), \qquad \alpha(t) \text{ non-decreasing}, \quad n = 0, 1, 2, \cdots, \tag{1.2}$$

was set and solved by T. J. Stieltjes. It would be natural to attempt to generalize the problem by requiring merely that $\alpha(t)$ should be a function of bounded variation on $(0, \infty)$; but the generalized problem has, as Theorem 1 shows, a trivial solution.

To establish Theorem 1, we shall exhibit an arbitrary real sequence $\{\mu_n\}$ as the difference of two sequences $\{\lambda_n\}$ and $\{\nu_n\}$, each of the form (1.2).† The construction will also lead to the result that any sequence $\{\mu^n\}$ of positive numbers of sufficiently rapid growth has the form (1.2); it is sufficient, for example, that

$$\mu_0 \geqq 1, \qquad \mu_n \geqq (n\mu_{n-1})^n, \qquad n \geqq 1. \tag{1.3}$$

A specimen sequence satisfying (1.3) is $\mu_0 = 1, \mu_n = n^{n^n}$, $(n = 1, 2, \cdots)$.

As an application‡ of Theorem 1, it will be shown that

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† Added in proof: Other proofs of Theorem 1 have been given by G. Pólya (*Sur l'indétermination d'un problème voisin du problème des moments*, Comptes Rendus de l'Académie des Sciences, Paris, vol. 207 (1938), pp. 708–711). Pólya points out that a theorem of which Theorem 1 is an immediate consequence was proved by É. Borel in 1894.

‡ For another application of Theorem 1, see J. Shohat, *Sur les polynomes orthogonaux généralisés*, Comptes Rendus de l'Académie des Sciences, Paris, vol. 207 (1938), pp 556–558.

$$f(x) = \int_0^\infty x(t)d\alpha(t),$$

with

$$\int_0^\infty t^n |\, d\alpha(t)\,| < \infty, \qquad n = 1, 2, \cdots,$$

is not the general linear functional on any very interesting space of functions $x = x(t)$, containing an infinite number of the functions t^n, $(n = 1, 2, \cdots)$ (see §4 for a precise statement). Other negative results of this character have been obtained by J. W. Tukey and the author;* the reader is referred to their paper for a discussion of the significance of such results.

2. **Proof of Theorem 1.** We use the notation

$$[\mu_0\mu_2 \cdots \mu_{2n}] = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \cdot & \cdot & \cdots & \cdot \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix}, \qquad n = 0, 1, 2, \cdots.$$

Then a necessary and sufficient condition for $\{\mu_n\}$ to have the form 1.2 is†

$$(2.1) \qquad [\mu_0\mu_2 \cdots \mu_{2n}] \geqq 0, \quad [\mu_1\mu_3 \cdots \mu_{2n+1}] \geqq 0, \quad n = 0, 1, 2, \cdots.$$

We choose positive numbers λ_0, λ_1, ν_0, ν_1, so that $\lambda_0 - \nu_0 = \mu_0$, $\lambda_1 - \nu_1 = \mu_1$. We now proceed to define the sequences $\{\lambda_n\}$, $\{\nu_n\}$ by induction. Suppose that

$$(2.2) \qquad \lambda_k - \nu_k = \mu_k$$

for $k = 0, 1, 2, \cdots, 2n-1$, and that the determinants

$$(2.3) \qquad \begin{matrix} [\lambda_0\lambda_2 \cdots \lambda_{2k}], & [\nu_0\nu_2 \cdots \nu_{2k}], \\ [\lambda_1\lambda_3 \cdots \lambda_{2k+1}], & [\nu_1\nu_3 \cdots \nu_{2k+1}], \end{matrix}$$

are positive for $k = 0, 1, 2, \cdots, n-1$. We have (with undetermined λ_{2n})

$$[\lambda_0\lambda_2 \cdots \lambda_{2n}] = \lambda_{2n}[\lambda_0\lambda_2 \cdots \lambda_{2n-2}] + P,$$

where P is a polynomial in $\lambda_0, \lambda_1, \cdots, \lambda_{2n-1}$; and there is a corre-

* R. P. Boas, Jr., and J. W. Tukey, *A note on linear functionals*, this Bulletin, vol. 44 (1938), pp. 523–528.

† See, for example, O. Perron, *Die Lehre von den Kettenbrüchen*, 1929, p. 410; cf. also M. Riesz, *Sur le problème des moments, troisième note*, Arkiv för Matematik, Astronomi och Fysik, vol. 17 (1922–1923), no. 16.

sponding relation for $[\nu_0\nu_2\cdots\nu_{2n}]$. Since $[\lambda_0\lambda_2\cdots\lambda_{2n-2}]>0$, and $[\nu_0\nu_2\cdots\nu_{2n-2}]>0$, we can choose λ_{2n} and ν_{2n} so that $\lambda_{2n}-\nu_{2n}=\mu_{2n}$, and so large that $[\lambda_0\lambda_2\cdots\lambda_{2n}]>0$, $[\nu_0\nu_2\cdots\nu_{2n}]>0$. Similarly we can then choose λ_{2n+1} and ν_{2n+1} so that $\lambda_{2n+1}-\nu_{2n+1}=\mu_{2n+1}$, $[\lambda_1\lambda_3\cdots\lambda_{2n+1}]>0$, $[\nu_1\nu_3\cdots\nu_{2n+1}]>0$. This completes the induction: we can find sequences $\{\lambda_n\}$, $\{\nu_n\}$ such that for $k=0, 1, 2, \cdots$, (2.2) is satisfied, and all the determinants (2.3) are positive. Then $\{\lambda_n\}$ and $\{\nu_n\}$ satisfy (2.1), and consequently have the form (1.2), so that $\{\mu_n\}$ has the form (1.1).

3. **Rapidly increasing sequences.** We now prove the following theorem:

THEOREM 2. *If*

$$\mu_0 \geqq 1, \qquad \mu_n \geqq (n\mu_{n-1})^n, \qquad n = 1, 2, \cdots, \tag{3.1}$$

then $\{\mu_n\}$ *has the form* (1.2).

For the proof, we modify the construction of the sequence $\{\lambda_n\}$ of §2. We have, for $n=1, 2, \cdots$,

$$[\mu_0\mu_2\cdots\mu_{2n}] = \mu_{2n}[\mu_0\mu_2\cdots\mu_{2n-2}] + \sum_{k=n}^{2n-1} \pm \mu_k D_k, \tag{3.2}$$

where the D_k are n-rowed minors of $[\mu_0\mu_2\cdots\mu_{2n}]$ and do not involve μ_{2n}. Similarly, for $n=1, 2, \cdots$,

$$[\mu_1\mu_3\cdots\mu_{2n+1}] = \mu_{2n+1}[\mu_1\mu_3\cdots\mu_{2n-1}] + \sum_{k=n+1}^{2n} \pm \mu_k D_k', \tag{3.3}$$

where the D_k' are n-rowed minors of $[\mu_1\mu_3\cdots\mu_{2n+1}]$, not involving μ_{2n+1}.

Suppose that for $k\leqq n-1$, $(n=1, 2, \cdots)$,

$$[\mu_0\mu_2\cdots\mu_{2k}] \geqq 1, \qquad [\mu_1\mu_3\cdots\mu_{2k+1}] \geqq 1. \tag{3.4}$$

Assuming (3.1), we shall show that (3.4) is satisfied also for $k=n$.

Clearly, $\mu_m\geqq 1$ for $m=1, 2, \cdots$. Hence we have

$$\mu_m \geqq (m\mu_{m-1})^m > 2(m/2)^{(m+4)/4\ (m+2)/2}\mu_{m-1}, \quad m = 2, 3, \cdots.$$

Therefore

$$\mu_{2n} > 1 + n^{(n+2)/2\ n+1}\mu_{2n-1}, \qquad \mu_{2n+1} > 1 + n^{(n+2)/2\ n+1}\mu_{2n}. \tag{3.5}$$

Now, (3.1) implies in particular that $\mu_{m+1}\geqq\mu_m$, $(m\geqq 1)$; hence the elements of the determinants D_k do not exceed μ_{2n-1}, and the ele-

ments of the D_k' do not exceed μ_{2n}. Then by Hadamard's theorem,*

$$|D_k| \leqq \mu_{2n-1}^{n} n^{n/2}, \qquad k = n, n+1, \cdots, 2n-1,$$

$$|D_k'| \leqq \mu_{2n}^{n} n^{n/2}, \qquad k = n+1, n+2, \cdots, 2n.$$

Therefore, using (3.2), (3.3), (3.4), (3.5), we obtain

$$[\mu_0\mu_2 \cdots \mu_{2n}] \geqq \mu_{2n} - n^{1+n/2}\, \mu_{2n-1}^{n+1} > 1,$$

$$[\mu_1\mu_3 \cdots \mu_{2n+1}] \geqq \mu_{2n+1} - n^{1+n/2}\, \mu_{2n}^{n+1} > 1.$$

Thus (3.4) holds for $k=n$ if it holds for $k<n$; but it holds for $k=0$ by assumption, and consequently holds for all k; therefore $\{\mu_n\}$ has the form (1.2).

The moment problem (1.2) is said to be determined or undetermined according as the function $\alpha(t)$ is or is not unique (after being normalized by the conditions $\alpha(0)=0$, $\alpha(t)=[\alpha(t+)+\alpha(t-)]/2$ for $t>0$). A consequence of Theorem 2 is that the moment problem (1.2) is not only solvable for any sequence $\{\mu_n\}$ of sufficiently rapid growth, but is even undetermined. In fact, if $\{\mu_n\}$ satisfies (3.1) and if in addition $\mu_2 \geqq (2\mu_1+2)^2$, we define a sequence $\{\nu_n\}$ by setting $\nu_1=\mu_1+1$, $\nu_n=\mu_n$ for $n\neq 1$. Then $\{\nu_n\}$ satisfies (3.1); consequently for $n=0, 1, 2, \cdots$,

$$\nu_{2n} = \int_0^\infty t^{2n} d\beta(t) = \int_0^\infty u^n d\beta(u^{1/2}) = \int_0^\infty u^n d\gamma(u),$$

say; while

$$\nu_{2n} = \mu_{2n} = \int_0^\infty t^{2n} d\alpha(t) = \int_0^\infty u^n d\delta(u),$$

where $\gamma(u)$ and $\delta(u)$ are normalized and non-decreasing. But $\gamma(u)$ and $\delta(u)$ are distinct, since

$$\nu_1 = \int_0^\infty u^{1/2} d\gamma(u) = 1 + \int_0^\infty u^{1/2} d\delta(u) = 1 + \mu_1.$$

Hence the moment problem for the sequence $\{\mu_{2n}\}$ is undetermined.

4. **Linear functionals.** We use the terminology of S. Banach's book.† Let R be a topological vector space of elements x, let P be a

* G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, 1934, p. 34.

† *Théorie des Opérations Linéaires*, 1932.

space of elements p, and let $f_p(x)$ be a functional with domain R, defined for each p in P. We say that a general linear functional in R is $f_p(x)$, if the following conditions are satisfied:

(i) *$f_p(x)$ is a linear functional for every $p \in P$.*

(ii) *Every linear functional $g(x)$ with domain R is identically equal to some $f_p(x)$.*

In the application to be made here, the elements of P are the functions $p=p(t)$, of bounded variation on $(0, \infty)$, such that

$$\int_0^\infty t^n |dp(t)| < \infty, \qquad n=1, 2, \cdots;$$

the elements of R are measurable functions $x=x(t)$, defined on $(0, \infty)$; and

$$f_p(x) = \int_0^\infty x(t)dp(t), \tag{4.1}$$

where the integral is a Lebesgue-Stieltjes integral. We have the following result:

THEOREM 3. *Let R be a topological vector space with the following property:**

(Q): *If $x \in R$ and $a_n \to 0$, then $a_n x \to \Theta$.†*

Then if R contains an infinite number of functions t^n, $(n=0, 1, 2, \cdots)$, there is some $p \in P$ for which (4.1) *is not a linear functional on R.*

In particular, we see that, under the hypotheses of Theorem 3, (4.1) is not a general linear functional on R.

Suppose that (4.1) is, for every $p \in P$, a linear functional on a space R with the specified properties. Let S be the subspace composed of all finite linear combinations of the elements t^n which are in R (with the topology of R). If f is an arbitrary distributive (that is, additive and homogeneous) functional with domain S, we define a sequence $\{\mu_n\}$ by setting $\mu_n = f(t^n)$ when $t^n \in R$, and $\mu_n = 0$ otherwise. By Theorem 1, there is a $p \in P$ such that

$$\mu_n = \int_0^\infty t^n dp(t), \qquad n = 0, 1, 2, \cdots.$$

Since f is distributive, we then have

* In particular, a space of type F has this property.

† Θ denotes the zero element of R.

$$f(x) = \int_0^\infty x(t)dp(t), \qquad x \in S. \tag{4.2}$$

Now (4.1) is a linear functional on R, and consequently a linear functional on S. Hence (4.2) states that every distributive functional on S is linear; but this is impossible unless S is finite-dimensional,* which it is not. This contradiction establishes the theorem.

NORTON, MASSACHUSETTS

ON FUNDAMENTAL SYSTEMS OF SYMMETRIC FUNCTIONS†

H. T. ENGSTROM

A set S of n polynomials over a field K, symmetric in n variables, $x_1, x_2, \cdots, x_n$, is said to form a fundamental system if any rational function over K, symmetric in these variables, can be expressed rationally in terms of the polynomials of S. In this paper we show that any n algebraically independent symmetric polynomials over a field K of characteristic zero form a fundamental system if the product of their degrees is less than $2n!$.

The result follows from a theorem due to Perron:‡

THEOREM 1. *Between $n+1$ polynomials (not constant), $f_1, f_2, \cdots, f_{n+1}$, in n variables, of degrees $m_1, m_2, \cdots, m_{n+1}$, respectively, there is always an identity of the form*

$$\sum C_{\nu_1\nu_2\cdots\nu_{n+1}} f_1^{\nu_1} f_2^{\nu_2} \cdots f_{n+1}^{\nu_{n+1}} \equiv 0,$$

where in each term,

$$\sum_{i=1}^{n+1} m_i\nu_i \leqq \prod_{i=1}^{n+1} m_i.$$

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* Let every distributive functional on S be linear, where S is a topological vector space with the property (Q). If S is infinite dimensional, let $\{x_n\}$, $(n=1, 2, \cdots)$, be an infinite set of linearly independent elements. Since $\lim_{k\to\infty} k^{-1}x_n = \Theta$, we can choose $y_n \in S$, $(n=1, 2, \cdots)$, linearly independent, with $y_n \to \Theta$. We set $f(y_n)=1$, $f(x)=0$ when x is not a finite linear combination of the y_n, $f(ax+by)=af(x)+bf(y)$ for any $x \in S$, $y \in S$; then f is a distributive functional on S, and hence is linear on S. Since $y_n \to \Theta$, $f(y_n) \to 0$ as $n \to \infty$; but this contradicts $f(y_n)=1$. Consequently S is finite dimensional.

† Presented to the Society, February 25, 1939, under the title *A note on fundamental systems of symmetric functions.*

‡ O. Perron, *Bemerkung zur Algebra*, Sitzungsberichte der Bayerischen Akademie, mathematisch-naturwissenschaftliche Abteilung, 1924, pp. 87–101.

The coefficients $C_{\nu_1\nu_2\cdots\nu_{n+1}}$ belong to the coefficient field of $f_1, f_2, \cdots, f_{n+1}$.

Consider any n algebraically independent polynomials $\phi_1, \phi_2, \cdots, \phi_n$, of degrees $m_1, m_2, \cdots, m_n$, with coefficients in a field K of characteristic zero. By Theorem 1 there exist relations

$$\Phi_i(x_i, \phi_1, \phi_2, \cdots, \phi_n) \equiv 0, \qquad i = 1, 2, \cdots, n, \tag{1}$$

each of degree less than or equal to $\prod_{i=1}^{n} m_i$ in x_i. The algebraic independence assures the actual presence of x_i in (1). It follows from (1) that the field $K(x_1, x_2, \cdots, x_n)$ of all rational functions of the $x_1, x_2, \cdots, x_n$ is a finite algebraic extension of the field $K(\phi_1, \phi_2, \cdots, \phi_n)$ generated by $\phi_1, \phi_2, \cdots, \phi_n$. Since K is of characteristic zero, this extension contains a primitive element ξ, which, by Theorem 1, satisfies a relation of the type (1) of degree less than or equal to $\prod_{i=1}^{n} m_i$ in ξ. Hence we have the following lemma:

LEMMA 1. *If $\phi_1, \phi_2, \cdots, \phi_n$ are n algebraically independent polynomials of degrees $m_1, m_2, \cdots, m_n$, then the field $K(x_1, x_2, \cdots, x_n)$ is a finite algebraic extension of $K(\phi_1, \phi_2, \cdots, \phi_n)$ of degree less than or equal to $\prod_{i=1}^{n} m_i$.*

The following result, which we state as a lemma, is well known:*

LEMMA 2. *If $a_1, a_2, \cdots, a_n$ are the elementary symmetric functions of $x_1, x_2, \cdots, x_n$, then $K(x_1, x_2, \cdots, x_n)$ is a Galois extension of $K(a_1, a_2, \cdots, a_n)$ of degree $n!$.*

Suppose now that $\phi_1, \phi_2, \cdots, \phi_n$ are algebraically independent and symmetric. Since $a_1, a_2, \cdots, a_n$ form a fundamental system of symmetric functions, it is clear that $K(a_1, a_2, \cdots, a_n)$ contains $K(\phi_1, \phi_2, \cdots, \phi_n)$. Hence the degree of $K(x_1, x_2, \cdots, x_n)$ over $K(\phi_1, \phi_2, \cdots, \phi_n)$ must be a multiple of the degree of $K(x_1, x_2, \cdots, x_n)$ over $K(a_1, a_2, \cdots, a_n)$. If $\prod_{i=1}^{n} m_i < 2n!$, it follows from Lemma 1 that the degree of $K(x_1, x_2, \cdots, x_n)$ over $K(\phi_1, \phi_2, \cdots, \phi_n)$ must be $n!$. Hence

$$K(\phi_1, \phi_2, \cdots, \phi_n) = K(a_1, a_2, \cdots, a_n),$$

and we have the theorem:

THEOREM 2. *Any set of n algebraically independent polynomials $\phi_1, \phi_2, \cdots, \phi_n$, symmetric in $x_1, x_2, \cdots, x_n$, over a field of characteristic zero forms a fundamental system if the product of their degrees is less than $2n!$.*

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* Cf. van der Waerden, *Moderne Algebra*, vol. 1, p. 173.

Theorem 2 is the best possible theorem of its kind; that is, the best general sufficiency condition for a fundamental system in terms of an upper bound for the product of the degrees without reference to the form of the polynomials $\phi_1, \phi_2, \cdots, \phi_n$. This may be verified by the example $\phi_1 = a_2$, $\phi_i = S_i$, $(i \geqq 2)$, where a_2 is the elementary symmetric function of degree 2, and S_i is the sum of the ith powers of the variables. In this case, the product of the degrees is $2n!$. The independence of $\phi_1, \phi_2, \cdots, \phi_n$ is established by showing the nonvanishing of the functional determinant D. The expression for D is

$$D = n! \cdot \begin{vmatrix} a_1 - x_1 & a_1 - x_2 & \cdots & a_1 - x_n \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \cdot & \cdot & \cdots & \cdot \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}$$

where $a_1 = x_1 + x_2 + \cdots + x_n$. After adding the second row to the first, and factoring a_1 from the first row, we have the Vandermonde determinant. Hence D does not vanish identically. On the other hand, $a_1 = (\phi_2 + 2\phi_1)^{1/2}$ is an irrational expression for a_1 whose uniqueness is guaranteed by the independence. In other words, a_1 cannot be expressed rationally in terms of the set $\phi_1, \phi_2, \cdots, \phi_n$, and the latter set does not form a fundamental system.

YALE UNIVERSITY

ON GREEN'S FUNCTIONS IN THE THEORY OF HEAT CONDUCTION*

H. S. CARSLAW AND J. C. JAEGER

1. **Introduction.** In this Bulletin (vol. 44 (1938), p. 125) Lowan discusses the Green's function for a line source at (r', θ') for the cases where the solid (i) is an infinite cylinder $r=a$, and (ii) is bounded internally by $r=a$, radiation taking place at $r=a$ into a medium at zero temperature.

He uses the method of the Laplace transformation. The solution for the first problem agrees with that already obtained by contour integration.† There are some obvious misprints in his discussion of the second problem, and in his solution on page 133 he seems to have used as boundary condition $G=0$ on $r=a$, instead of $\partial G/\partial r+hG=0$, in his notation. His result on that page should read

$$G = u + v = \frac{1}{4\pi} \sum_{n=-\infty}^{\infty} \cos n(\theta - \theta_0) \int_{-\infty}^{\infty} \alpha e^{-k\alpha^2 t} \frac{H_n^{(1)}(\alpha r_0)}{U_n(\alpha a)} \cdot \left\{ J_n(\alpha r) U_n(\alpha a) - H_n^{(1)}(\alpha r) \left[\alpha \frac{d}{dz} J_n(z) + h J_n(z) \right]_{z=\alpha a} \right\} d\alpha,$$

where

$$U_n(\alpha a) = \left[\alpha \frac{d}{dz} H_n^{(1)}(z) + h H_n^{(1)}(z) \right]_{z=\alpha a}.$$

Put in this form it can be reduced to the simpler form given below in (16), except for the difference in the sign of h.

In this paper we discuss this second problem, first by contour integration, and second by the Laplace transform. In the latter we use what appears to us a much simpler notation and a more rapid approach to the solution.

We remark also that we have used this notation and method in a number of other questions‡ and believe that it will be found increasingly useful and much simpler than the operational methods,

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* Presented to the Society, September 6, 1938.

† Carslaw, *Conduction of Heat*, 2d edition, 1921, §§88, 89. This book will be referred to below as C.H.

‡ Cf. Carslaw, *Operational methods in mathematical physics*, Mathematical Gazette, vol. 22 (1938), pp. 264–280; Carslaw and Jaeger, *Some problems in the mathematical theory of the conduction of heat*, Philosophical Magazine, (7), vol. 26 (1938), pp. 473–495.

due to Heaviside and Bromwich, expounded in Jeffreys' well known tract.*

2. **Solution using contour integration.** We take the line source at $(r', 0)$, and we require the solution of the equation

$$\frac{\partial v}{\partial t} = k\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}\right), \qquad t > 0,\ r > a, \tag{1}$$

which shall tend to zero when $t \to 0$ in $r > a$, except at $(r', 0)$, where it is to be infinite as

$$(4\pi kt)^{-1} \exp\left\{-(r^2 + r'^2 - 2rr' \cos\theta)/4kt\right\}. \tag{2}$$

Also, when $r \to a$, v is to satisfy

$$-\frac{\partial v}{\partial r} + hv = 0, \qquad t > 0. \tag{3}$$

Let

$$u = (4\pi kt)^{-1} \exp\left\{-(r^2 + r'^2 - 2rr' \cos\theta)/4kt\right\}. \tag{4}$$

Put $v = u + w$. Then w is to satisfy

$$\frac{\partial w}{\partial t} = k\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right), \qquad t > 0,\ r > a, \tag{5}$$

$$\lim_{t \to 0} w = 0, \qquad r > a, \tag{6}$$

$$-\frac{\partial w}{\partial r} + hw = \frac{\partial u}{\partial r} - hu, \qquad r = a,\ t > 0. \tag{7}$$

It is known† that

$$u = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \cos n\theta \int_0^{\infty} \alpha e^{-k\alpha^2 t} J_n(\alpha r') J_n(\alpha r) d\alpha \tag{8}$$

$$= -\frac{1}{4\pi}\sum_{n=-\infty}^{\infty} \cos n\theta \int \alpha e^{-k\alpha^2 t} J_n(\alpha r') H_n^{(1)}(\alpha r) d\alpha, \quad r > r', \tag{9}$$

$$= -\frac{1}{4\pi}\sum_{n=-\infty}^{\infty} \cos n\theta \int \alpha e^{-k\alpha^2 t} J_n(\alpha r) H_n^{(1)}(\alpha r') d\alpha, \quad r < r', \tag{10}$$

the integrals being taken over the path P of Fig. 1 in the α-plane.

* *Operational Methods in Mathematical Physics*, Cambridge Tracts in Mathematics and Mathematical Physics, no. 23.

† C.H., pp. 184–185.

For the path at infinity on the right the argument of 4α is less than π, and on the left it is greater than 3π. Let

$$(11) \qquad 4\pi w = \sum_{n=-\infty}^{\infty} \cos n\theta \int \alpha e^{-k\alpha^2 t} A_n(\alpha) H_n^{(1)}(\alpha r') H_n^{(1)}(\alpha r) d\alpha,$$

where $A_n(\alpha)$ is to be chosen so that w satisfies (7). Thus, using (10), we have

$$(12) \qquad A_n(\alpha)[\alpha H_n^{(1)\prime}(\alpha a) - hH_n^{(1)}(\alpha a)] = \alpha J_n'(\alpha a) - hJ_n(\alpha a).$$

Also

$$(13) \quad 4\pi v = \sum_{n=-\infty}^{\infty} \cos n\theta \int \alpha e^{-k\alpha^2 t} H_n^{(1)}(\alpha r')[A_n(\alpha) H_n^{(1)}(\alpha r) - J_n(\alpha r)] d\alpha,$$

when $r<r'$, and we interchange r and r' when $r>r'$.

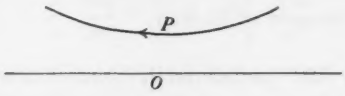

FIG. 1

This value of v satisfies (1) and (3). We shall now show that it satisfies (2); in other words, we shall show that $\lim_{t\to 0} w = 0$, when $r>a$.

The integral in (11) is continuous when $t \geqq 0$. We show that it vanishes when $t=0$. Take the closed circuit of Fig. 2, consisting of

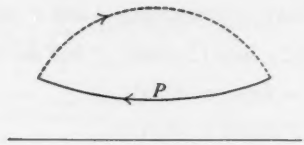

FIG. 2

the path P and the part of a circle, center at the origin, lying above the path P.

There are no zeros of $\alpha H_n^{(1)\prime}(\alpha a) - hH_n^{(1)}(\alpha a)$ for which the imaginary part of α is positive or zero.* Further the asymptotic expansions for the Bessel functions show that the integral

* See the footnote to §6.

(14) $$\int \alpha H_n^{(1)}(\alpha r) H_n^{(1)}(\alpha r') A_n(\alpha) d\alpha$$

over the circular arc above P tends to zero when the radius tends to infinity. It follows that the integral over the path P tends to zero. We have thus in (14) obtained a solution in the form of an integral over the path P.

3. Consider the closed circuit of Fig. 3, formed by the real axis, the path P, and the arcs of a circle with center at the origin and radius R (which will tend to ∞) joining the real axis and the path P.

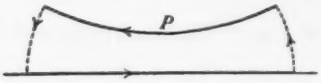

FIG. 3

When $t>0$, the integral of §2, (13),

$$\int \alpha e^{-k\alpha^2 t} H_n^{(1)}(\alpha r') [A_n(\alpha) H_n^{(1)}(\alpha r) - J_n(\alpha r)] d\alpha$$

over the circular arcs tends to zero when $R \to \infty$, r being less than r'. Also there are no poles of the integrand inside this circuit. It follows that the integral over the path P is equal to minus the integral from $-\infty$ to ∞ over the real axis. Thus we have

(15) $$4\pi v = \sum_{n=-\infty}^{\infty} \cos n\theta \int_{-\infty}^{\infty} \alpha e^{-k\alpha^2 t} H_n^{(1)}(\alpha r') [J_n(\alpha r) - A_n(\alpha) H_n^{(1)}(\alpha r)] d\alpha$$

when $r<r'$; and we have to interchange r and r' when $r>r'$.

4. If we break up $\int_{-\infty}^{\infty}$ into $\int_{-\infty}^{0}$ and $\int_{0}^{\infty}$ and use the formulas

$$H_n^{(1)}(x) = J_n(x) + iY_n(x),$$
$$H_n^{(1)\prime}(x) = J_n'(x) + iY_n'(x),$$
$$e^{in\pi} H_n^{(1)}(xe^{i\pi}) = H_n^{(1)}(x) - 2J_n(x) = -J_n(x) + iY_n(x),$$
$$e^{in\pi} H_n^{(1)\prime}(xe^{i\pi}) = J_n'(x) - iY_n'(x),$$

the result in §3, (15), reduces to

(16) $$v = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \cos n\theta \int_0^{\infty} \alpha e^{-k\alpha^2 t} \frac{U_n(\alpha r) U_n(\alpha r')}{[\alpha J_n'(\alpha a) - hJ_n(\alpha a)]^2 + [\alpha Y_n'(\alpha a) - hY_n(\alpha a)]^2} d\alpha,$$

where

$$U_n(\alpha r) = J_n(\alpha r)\left[\alpha Y_n'(\alpha a) - hY_n(\alpha a)\right] - Y_n(\alpha r)\left[\alpha J_n'(\alpha a) - hJ_n(\alpha a)\right];$$

and this, being symmetrical in r and r', holds both for $r<r'$ and $r>r'$.

5. **Solution using the Laplace transformation.** We start from equations (1) to (7) of §2; by equation (8), u is given in the form $u=\sum_{n=-\infty}^{\infty} u_n \cos n\theta$; so we seek a solution of type

$$v = u + w = \sum_{n=-\infty}^{\infty} (u_n + w_n)\cos n\theta, \tag{17}$$

where

$$\lim_{t\to 0} w_n = 0, \qquad r > a, \tag{18}$$

$$\frac{\partial^2 w_n}{\partial r^2} + \frac{1}{r}\frac{\partial w_n}{\partial r} - \frac{n^2}{r^2} w_n - \frac{1}{k}\frac{\partial w_n}{\partial t} = 0, \tag{19}$$

and the boundary condition (3) is satisfied by (u_n+w_n). Now apply the Laplace transformation, using $\bar{u}$, $\bar{v}$, $\cdots$ for the Laplace transforms of u, v, and so on. Thus,* writing $q=(p/k)^{1/2}$, we have

$$\begin{aligned}
\bar{u} = \int_0^\infty e^{-pt} u\,dt &= \frac{1}{4\pi k}\int_0^\infty \exp\left[-pt - \frac{1}{4kt}(r^2 + r'^2 - 2rr'\cos\theta)^{1/2}\right]\frac{dt}{t} \\
&= \frac{1}{2\pi k} K_0\left[q(r^2 + r'^2 - 2rr'\cos\theta)^{1/2}\right] \\
&= \begin{cases} \dfrac{1}{2\pi k}\displaystyle\sum_{n=-\infty}^{\infty} I_n(qr)K_n(qr')\cos n\theta, & r < r', \\ \dfrac{1}{2\pi k}\displaystyle\sum_{n=-\infty}^{\infty} I_n(qr')K_n(qr)\cos n\theta, & r > r', \end{cases} \qquad (20) \\
&= \sum_{n=-\infty}^{\infty} \bar{u}_n \cos n\theta,
\end{aligned}$$

where the $\bar{u}_n$ are given by (20). Also from (18) and (19)

$$\frac{d^2\bar{w}_n}{dr^2} + \frac{1}{r}\frac{d\bar{w}_n}{dr} - \left(\frac{n^2}{r^2} + q^2\right)\bar{w}_n = 0.$$

* We use successively the results, §6.22, (15), and §11.41, (8), of Watson, *Theory of Bessel Functions*.

So $\bar{w}_n = B_n(q)K_n(qr)$, where $B_n(q)$ is to be determined from the surface condition (3). This requires

$$-\frac{\partial \bar{v}}{\partial r} + h\bar{v} = 0, \qquad r = a.$$

Hence, for $r < r'$,

$$2\pi k B_n(q)\left[hK_n(qa) - qK_n'(qa)\right] = \left[qI_n'(qa) - hI_n(qa)\right]K_n(qr'). \tag{21}$$

So

$$\bar{v}_n = \bar{u}_n + \bar{w}_n = (1/2\pi k)K_n(qr')I_n(qr) + B_n(q)K_n(qr), \qquad r < r'.$$

Therefore, by the inversion formula,* if $\mu = (\lambda/k)^{1/2}$, and $c > 0$,

$$4\pi^2 ikv_n = \int_{c-i\infty}^{c+i\infty} e^{\lambda t}K_n(\mu r')\left[I_n(\mu r) + 2\pi k B_n(\mu)K_n(\mu r)\right]d\lambda, \quad r < r', \tag{22}$$

and

$$v = \sum_{n=-\infty}^{\infty} v_n \cos n\theta. \tag{23}$$

We have to verify that this value of v satisfies (2); that is, that the w_n satisfy (18). From (21) and the inversion formula, we obtain

$$2\pi i w_n = \int_{c-i\infty}^{c+i\infty} e^{\lambda t}K_n(\mu r)B_n(\mu)d\lambda,$$

and this is a continuous function of t for $t \geqq 0$. To show that it vanishes for $t = 0$, consider the integral

$$\int K_n(\mu r)B_n(\mu)d\lambda \tag{24}$$

taken round the contour $ABGA$ of Fig. 4, consisting of the line joining $c - iR$, $c + iR$, and a portion of a circle, center the origin, in the right-hand half-plane. There are no poles of the integrand in the contour.† Also, using the asymptotic expansions for the Bessel functions, we find that the integral in (24) round the circular arc tends to zero as $R \to \infty$. So the integral over the path $c - i\infty$, $c + i\infty$ vanishes.

6. To reduce (22) to the form (16), consider the integral of the integrand of (22) round the path of Fig. 4 formed by the line AB at distance c from the imaginary axis, the circle Γ of radius R, the circle γ of radius ϵ, and the lines CD, EF, on which $\arg \lambda = \pi$, and $-\pi$, respectively. There are no poles of the integrand inside or on the

* Churchill, Mathematische Zeitschrift, vol. 42 (1937), p. 569.

† See the footnote to §6.

contour.* The integral round γ tends to zero as $\epsilon \to 0$. Also, using the asymptotic expansions of the Bessel functions, we find that the integral round Γ tends to zero as $R \to \infty$. So in the limit $R \to \infty$, $\epsilon \to 0$,

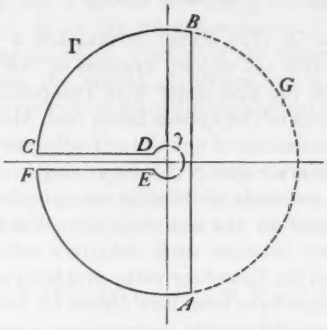

FIG. 4

v_n equals the sum of the integrals along CD and EF. Putting $\lambda = k\alpha^2 e^{i\pi}$ in the former, and $\lambda = k\alpha^2 e^{-i\pi}$ in the latter, we obtain, on reduction, (16).

7. The path P of Fig. 1 in the α-plane corresponds to a path in the λ-plane ($\lambda = -k\alpha^2$) which starts at infinity where arg λ lies between $-\pi$ and $-\pi/2$, passes to the right of the origin, and ends at infinity, where arg λ lies between $\pi/2$ and π. This is the path L' of Jeffreys' tract, *Operational Methods in Mathematical Physics* (2d edition, 1931, p. 29) and not the path L (from $c-i\infty$ to $c+i\infty$) as stated by Bromwich in the Proceedings of the Cambridge Philosophical Society, vol. 20 (1921), p. 412. The connection between the two methods employed in this paper is thus clear. There may be an advantage in the former as it seems to give a simpler verification that the expression as found does in fact satisfy all the conditions of the problem.

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* To show that $zK_n'(z) - bK_n(z)$, where $b>0$, has no zeros for $R(z) \geqq 0$, we may, since $K_{-n}(z) = K_n(z)$, take $n \geqq 0$. That there are no real positive zeros follows from the recurrence formula and the fact that $K_n(x) > 0$ for real positive x. That there are no complex zeros follows from the formula (Gray and Matthews, *Treatise on Bessel Functions*, 2d edition, 1922, p. 70, (30)): $(\lambda^2-\mu^2)\int_a^\infty xK_n(\lambda x)K_n(\mu x)dx = a[\mu K_n(\lambda a)K_n'(\mu a) - \lambda K_n(\mu a)K_n'(\lambda a)]$, $R(\lambda+\mu) > 0$. To show there are no pure imaginary zeros $z = iy$, we have $iyK_n'(iy) - bK_n(iy) = \frac{1}{2}\pi i e^{3n i \pi/2}[-yJ_n'(y) + bJ_n(y) + iyY_n'(y) - ibY_n(y)]$, and the real and imaginary parts of this must vanish. This requires $J_n(y)Y_n'(y) - Y_n(y)J_n'(y) = 0$, but it is $2/\pi y$.

SOME REMARKS ON LINEAR DIFFERENTIAL SYSTEMS*

WILLIAM T. REID

1. **Introduction.** In 1926 Bliss† formulated a definition of self-adjoint and definitely self-adjoint systems of ordinary linear differential equations of the first order with two-point boundary conditions, the coefficients of the system being real. More recently, Bliss‡ has modified the definition of definite self-adjointness in such a manner that the boundary value problems arising in a well known way from nonsingular problems of Bolza in the calculus of variations are definitely self-adjoint by the new definition. Further intimate relationships that exist between such definitely self-adjoint boundary value problems and the boundary value problems associated with the calculus of variations have been established by Reid.§

In this note linear differential systems with complex-valued coefficients are considered and self-conjugate adjoint and definitely self-conjugate adjoint systems are defined. It is shown that definitely self-conjugate adjoint systems have only real characteristic values, and that for a consideration of the questions of the existence of characteristic values and associated expansion theorems such a system may be replaced by a corresponding definitely self-adjoint system with real coefficients. In §4 there is solved an associated matrix differential equation, and the question of self-conjugate adjointness is reduced to the determination of a nonsingular constant matrix of a certain type. Finally, there is given an application of this matrix differential equation to the theory of matrix differential equations of Riccati type.

Matrix notation is used throughout. Capital italic letters denote n-rowed square matrices whose elements are complex-valued, the element in the ith row and jth column being denoted by the same letter with the subscript ij. Lower case italic letters signify vectors with n components, the ith component being denoted by a subscript i. If $M \equiv \| M_{ij} \|$, $u \equiv [u_i]$, the vectors $[M_{ij}u_j]$ and $[u_jM_{ji}]$ are denoted by Mu and uM, respectively, where the repetition of the subscript j indi-

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* Presented to the Society, September 6, 1938.

† Transactions of this Society, vol. 28 (1926), pp. 561–584. This paper will be referred to as B.I.

‡ Transactions of this Society, vol. 44 (1938), pp. 413–428. This paper will be referred to as B.II.

§ Transactions of this Society, vol. 44 (1938), pp. 508–520.

cates summation with respect to this subscript over the range $1, \cdots, n$. The scalar product $u_i v_j$ of two vectors is denoted simply by uv. If α is a scalar, $\bar{\alpha}$ is its complex conjugate, and for a vector u we write $\bar{u}$ for $[\bar{u}_i]$. For a matrix $M \equiv \|M_{ij}\|$ we use $\overline{M}$ for the conjugate matrix $\|\overline{M}_{ij}\|$, $\tilde{M}$ for the transpose matrix $\|M_{ji}\|$, and M^* for the conjugate transpose $\|\overline{M}_{ji}\|$. Finally, if the elements of M are differentiable functions, the matrix of derivatives is denoted by M'; similarly, if the components of u are differentiable functions, we write $u' \equiv [u_i']$.

2. **Self-conjugate adjoint systems.** Using the notation of the preceding paragraph, the differential system which we shall consider involves the vector differential equations and the two-point boundary conditions

$$(2.1) \qquad y' = [A(x) + \lambda B(x)]y, \qquad s[y] \equiv My(a) + Ny(b) = 0.$$

It will be supposed that the elements of A and B are continuous functions of the real variable x on the finite interval ab: $a \leqq x \leqq b$, and that the constant matrices M and N are such that the relations $s_i[y]$ are linearly independent. If λ is such that (2.1) has a solution $y(x) \not\equiv 0$ on ab, λ is called a *characteristic value*, and such a solution is termed a *characteristic solution* of (2.1). If $p \equiv [p_i] \equiv [P_{ij}]$, $q \equiv [q_i] \equiv [Q_{ij}]$, $(j = 1, \cdots, n)$, form n linearly independent solutions of the linear equations $Mp - Nq = 0$, the system adjoint to (2.1) is (see B.I, p. 565),

$$(2.2) \qquad z' = -z[A(x) + \lambda B(x)], \qquad t[z] \equiv z(a)P + z(b)Q = 0.$$

The system

$$(2.3) \qquad u' = -u[\overline{A}(x) + \lambda \overline{B}(x)], \qquad \bar{t}[u] \equiv u(a)\overline{P} + u(b)\overline{Q} = 0$$

we shall term the *conjugate* of (2.2), and the *conjugate adjoint* of (2.1). Clearly if the elements of $A(x)$, $B(x)$, M, and N are real, the elements of P and Q may be chosen real, and in this case system (2.3) is identical with (2.2). In general, if $u = u(x)$ is a solution of (2.3) for a value λ, then $z = \bar{u}(x)$ is a solution of (2.2) for the corresponding conjugate value $\bar{\lambda}$.

The system (2.1) will be called *self-conjugate adjoint* if it is equivalent to (2.3) under a linear transformation $u = T(x)y$, where the elements of T are complex-valued functions which are continuous and have continuous derivatives on ab, and the determinant of T is different from zero on this interval. For brevity, a nonsingular matrix $T(x)$ whose elements have continuous derivatives on ab will be termed an *admissible transformation matrix*.

THEOREM 2.1. *The system* (2.1) *is self-conjugate adjoint if and only if there is an admissible transformation matrix such that*

$$TA + A^*T + T' \equiv 0, \qquad TB + B^*T \equiv 0, \tag{2.4}$$

$$MT^{-1}(a)M^* = NT^{-1}(b)N^*. \tag{2.5}$$

Relations (2.4) follow immediately upon the substitution of $u = Ty$ in (2.3). In order for the linear equations $s[y] = 0$ to be equivalent to the set $\bar{l}[u] = 0$ under this transformation, it is seen to be necessary and sufficient that there exist a nonsingular constant matrix C such that

$$CP^*T(a) = M, \qquad CQ^*T(b) = N, \tag{2.6}$$

and since $P^*M^* - Q^*N^* = 0$, it is readily seen that (2.5) is equivalent to (2.6). If the coefficients of the system (2.1) are real and T is assumed to be real, conditions (2.4), (2.5) are of course exactly those obtained by Bliss (B.I, p. 569).

3. **Definitely self-conjugate adjoint systems.** Corresponding to the modified definition of definitely self-adjoint systems with real coefficients as given by Bliss in B.II, we say that the system (2.1) is *definitely self-conjugate adjoint* if there is an admissible transformation matrix T with respect to which it is self-conjugate adjoint, and the corresponding matrix $S(x) \equiv T^*(x)B(x)$ satisfies the conditions:

(i) *$S(x)$ is positive semi-definite and hermitian; that is, $S(x) \equiv S^*(x)$ and for arbitrary vectors y the form $\bar{y}S(x)y$ is nonnegative.*

(ii) *If $y(x) \not\equiv 0$ is a solution of system* (2.1) *for some value λ, then $\bar{y}(x)S(x)y(x) \not\equiv 0$ on ab.*

THEOREM 3.1. *All the characteristic values of a definitely self-conjugate adjoint system* (2.1) *are real.*

For if $y(x)$ is a characteristic solution of (2.1) corresponding to a characteristic value λ, then $z \equiv \bar{T}(x)\bar{y}(x)$ is a solution of (2.2) for the conjugate value $\bar{\lambda}$. From the differential equations of (2.1) and (2.2) we obtain

$$z(x)y(x)\Big|_a^b = (\lambda - \bar{\lambda}) \int_a^b z(x)B(x)y(x)dx.$$

Since $s[y] = 0$ and $t[z] = 0$, we also have $z(x)y(x)\big|_a^b = 0$.† If λ is not real, we then have

† See B.I, p. 565. Obviously the derivation of the formula (7) of B.I does not involve the reality of the elements of M and N.

$$0 = \int_a^b z(x)B(x)y(x)dx = \int_a^b \bar{y}(x)T^*(x)B(x)y(x)dx$$
$$= \int_a^b \bar{y}(x)S(x)y(x)dx,$$

which, in view of the above hypotheses (i) and (ii), is impossible. Hence all the characteristic values of (2.1) are real.

Without loss of generality one may replace in (2.1) the parameter λ by $\lambda\alpha$ where α is an arbitrary nonzero complex number. This is equivalent to replacing the matrix $B(x) \equiv \|B_{ij}(x)\|$ by $\alpha B(x) \equiv \|\alpha B_{ij}(x)\|$. We therefore have the following theorem:

THEOREM 3.2. *Suppose that there exists a nonzero complex number α and an admissible transformation matrix $T(x)$ satisfying*

$$(2.4_\alpha) \qquad TA + A^*T + T' \equiv 0, \qquad \alpha TB + \bar{\alpha}B^*T \equiv 0,$$

the condition (2.5), *and that the matrix $S_\alpha(x) = \alpha T^*B$ satisfies conditions* (i) *and* (ii) *of* §3. *Then all the nonzero characteristic values λ of* (2.1) *have* $\arg \lambda = \arg \alpha$.

As for the case of a definitely self-adjoint system with real coefficients (see B.II), one may proceed directly to show that for a definitely self-conjugate adjoint system the index of a characteristic value is equal to its multiplicity, and derive certain expansion theorems in terms of its characteristic solutions. In view of Theorem 3.1, however, the consideration of the existence of characteristic values and related expansion theorems for a definitely self-conjugate adjoint system (2.1) is reducible to the same consideration for an associated definitely self-adjoint system with real coefficients.

Corresponding to a square n-rowed matrix $K \equiv \|K_{ij}\| \equiv \|K_{ij}^1 + (-1)^{1/2}K_{ij}^2\|$ the corresponding script letter $\mathcal{K}$ shall denote the square $2n$-rowed matrix

$$(3.1) \qquad \mathcal{K} \equiv \begin{Vmatrix} K^1 & -K^2 \\ K^2 & K^1 \end{Vmatrix}$$

of real elements. Corresponding to a vector $y \equiv [y_i] \equiv [y_i^1 + (-1)^{1/2}y_i^2]$, η shall be defined by $\eta \equiv [y_1^1, \cdots, y_n^1, y_1^2, \cdots, y_n^2]$. Separating real and pure imaginary parts of (2.1) *for real values of* λ, we may write this system as

$$(3.2) \qquad \eta' = [\mathcal{A} + \lambda\mathcal{B}]\eta, \qquad \mathcal{M}\eta(a) + \mathcal{N}\eta(b) = 0.$$

If the system (2.1) is self-conjugate adjoint with respect to an ad-

missible transformation matrix $T(x)$, then the corresponding matrix $\mathfrak{T}(x)$ is an admissible transformation matrix with respect to which (3.2) is self-adjoint. Moreover, if (2.1) is definitely self-conjugate adjoint with respect to an admissible transformation matrix $T(x)$, then the real system (3.2) is definitely self-adjoint with respect to $\mathfrak{T}(x)$. The relations (2.4), (2.5) and conditions (i), (ii) may be verified directly. They are, however, immediate consequences of the fact that the algebra of square n-rowed matrices K with complex elements is equivalent to the algebra of square $2n$-rowed matrices $\mathfrak{K}$ of the form (3.1), with real elements.

4. **A special matrix differential equation.** In §2 we have shown that conditions (2.4) and (2.5) are necessary and sufficient for system (2.1) to be self-conjugate adjoint with respect to an admissible transformation matrix $T(x)$. The first relation of (2.4) is a matrix differential equation, and, in particular, $T(x)$ must be a solution of this equation with determinant different from zero on ab. This matrix differential equation may obviously be considered as a linear vector differential system in the n^2 elements $T_{ij}(x)$, $(i, j=1, \cdots, n)$. The condition that $|T_{ij}(x)| \neq 0$ on ab is, however, a property of a single solution of this equation. The result that we wish is contained as a special instance of the following theorem:

THEOREM 4.1. *Suppose the elements of the matrices $H(x)$, $K(x)$ are continuous functions of x on ab. Then the general solution $T(x)$ of the matrix differential equation*

$$T' = H(x)T + TK(x) \tag{4.1}$$

is of the form $T(x) = U(x)CV(x)$, where $U(x)$ is a fundamental matrix solution† of $U' = H(x)U$, $V(x)$ is a fundamental matrix solution of $V' = VK(x)$, and C is an arbitrary constant matrix. In particular, if $|T(x_0)| = 0$ for a point x_0 of ab, then $|T(x)| \equiv 0$ on ab.

To prove the general form of $T(x)$ we have merely to write $T(x) = U(x)W(x)$, where $U(x)$ is a fundamental matrix solution of $U' = H(x)U$. Upon substituting in (4.1) we obtain $UW' = UWK$, which, since $|U| \neq 0$ on ab, is equivalent to $W' = WK$. But the general solution of this last equation is $W = CV$, where C is a constant matrix and V is a fundamental matrix solution of $V' = VK(x)$. Since

$$|T(x)| = |U(x)| \cdot |C| \cdot |V(x)|$$

and $|U(x)| \neq 0$, $|V(x)| \neq 0$ on ab, the rest of the theorem is immediate.

† That is, a solution with $|U(x)| \neq 0$ on ab.

Applying this result to the matrix differential equation of (2.4), we have $H=-A^*$, $K=-A$. Moreover, it is seen that if $V(x)$ is a fundamental matrix solution of $V'=-VA(x)$, then $U=V^*(x)$ is a fundamental matrix solution of $U'=-A^*(x)U$. Hence the general form of the solution of this equation is $T(x)=V^*(x)CV(x)$. Consequently, the question of whether or not there is an admissible transformation matrix T with respect to which system (2.1) is self-conjugate adjoint is equivalent to the question of whether or not there is a nonsingular constant matrix C which satisfies with a fundamental matrix solution $V(x)$ of $V'=-VA(x)$ the conditions

$$
\begin{aligned}
&V^*(x)CV(x)B(x)+B^*(x)V^*(x)CV(x)\equiv 0,\\
&MV^{-1}(a)C^{-1}V^{*-1}(a)M^*=NV^{-1}(b)C^{-1}V^{*-1}(b)N^*.
\end{aligned}
\tag{4.2}
$$

In conclusion, we shall give an application of Theorem 4.1 to nonlinear matrix differential equations of the form

$$
Y'+YE(x)Y=R(x),
\tag{4.3}
$$

where the elements of $E(x)$ and $R(x)$ are continuous on the interval ab.

THEOREM 4.2. *If $Y=Y_1(x)$, $Y=Y_2(x)$ are solutions of* (4.3) *whose elements are continuous on the interval ab, and if $T(x)\equiv Y_1(x)-Y_2(x)$ is such that $|T(x_0)|=0$ at a point x_0 of ab, then $|T(x)|\equiv 0$ on ab.*

It may be verified directly that $T\equiv Y_1-Y_2$ satisfies (4.1) with

$$
\begin{aligned}
H(x)&=-(1/2)[Y_1(x)+Y_2(x)]E(x),\\
K(x)&=-(1/2)E(x)[Y_1(x)+Y_2(x)];
\end{aligned}
$$

hence Theorem 4.2 is an immediate consequence of Theorem 4.1.

The hypotheses of a theorem due to W. M. Whyburn† can be substantially weakened if use is made of the above theorem. Whyburn considered the matrix equation of Riccati type

$$
Y'+YY=R(x)
$$

and stated specifically‡ the assumption that the determinant of the difference of two solutions of this equation be different from zero on the entire interval considered. In view of the above theorem one may simply assume that this determinant is not identically zero on the interval.

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† American Journal of Mathematics, vol. 56 (1934), pp. 587–592.

‡ See loc. cit., Theorem 4, p. 590.

ON NON-BOUNDARY SETS*

A. D. WALLACE

The purpose of this note is largely methodological; namely, to complete the triology of dense, boundary, and nondense sets by adding non-boundary sets.

We adhere to the nomenclature of Kuratowski's *Topologie*† except as noted. In particular, we suppose that S is a nonvacuous space satisfying his axioms of closure, and we write $F(X)=\overline{X}\cdot\overline{CX}$ and $X^0=C\overline{CX}$ where $CX=S-X$. If the set X has the property P, we write X^P or $(X)^P$, and in the contrary case X^{CP} or $(X)^{CP}$.

A set is *dense* if its closure is the space; *boundary* if its complement is dense; *nondense* if its closure is boundary; and finally, *non-boundary* if its complement is nondense. We designate the properties by D, B, ND, and NB, respectively.

THEOREM. *The following conditions are necessary and sufficient in order that a set be*

I. *Dense: The interior of its closure is the space; the boundary of its complement is the closure of its complement; its complement is a boundary set; its closure is a non-boundary set.*

II. *A boundary set: The closure of its interior is null; its boundary is its closure; its complement is dense; its interior is nondense.*

III. *Nondense: The interior of its closure is null; the boundary of its closure is its closure; its complement is a non-boundary set; its closure is a boundary set.*

IV. *A non-boundary set: The closure of its interior is the space; the boundary of the closure of its complement is the closure of its complement; its complement is nondense; its interior is dense.*

We summarize this in the following table of equivalences. The Roman numerals correspond to the statements above, and each statement in a row is equivalent to every other statement in that row.

The proofs of these statements are as follows: Column 2 is a formulation of the definitions. In column 3 statement I 3 follows from I 2 since $S^0=S$; II 2 is equivalent to $X^0=0$, which is clearly the same as II 3; III 3 is the complement of III 2; IV 3 is IV 2.

As to column 4, we have for I 4

$$(\overline{X}=S)\rightarrow(\overline{X}\cdot\overline{CX}=\overline{CX})\rightarrow(F(CX)=\overline{CX}).$$

* Presented to the Society, February 25, 1939.

† C. Kuratowski, *Topologie* I, Warsaw, 1933.

Also $\overline{X}\cdot\overline{CX}=\overline{CX}$ is the same as saying that $\overline{CX}$ is a subset of $\overline{X}$. But $C\overline{X}\subset CX\subset\overline{CX}\subset\overline{X}$, or $C\overline{X}=0$. The remaining statements in this column follow from this one using only the definitions.

	1	2	3	4	5	6
I	X^D	$\overline{X}=S$	$\overline{X^0}=S$	$F(CX)=\overline{CX}$	$(CX)^B$	$(\overline{X})^{NB}$
II	X^B	$\overline{CX}=S$	$\overline{X^0}=0$	$F(X)=\overline{X}$	$(CX)^D$	$(X^0)^{ND}$
III	X^{ND}	$C\overline{X}=S$	$\overline{X^0}=0$	$F(\overline{X})=\overline{X}$	$(CX)^{NB}$	$(\overline{X})^B$
IV	X^{NB}	$\overline{C\overline{C}\overline{X}}=S$	$\overline{X^0}=S$	$F(\overline{CX})=\overline{CX}$	$(CX)^{ND}$	$(X^0)^D$

Column 5 can be deduced from column 4 and the definitions. Statement IV 6 is the same as IV 2; III 6 is II 2 with X replaced by $\overline{X}$; for II 6 we have

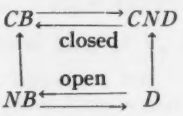
$$(X^0)^{ND}\equiv(\overline{CX^0}=S)\equiv(\overline{\overline{CX}}=S)\equiv X^B;$$

the proof of I 6 is similar.

We have the following theorem giving relations between the various properties:

THEOREM. *If a set is a non-boundary set, it is not a boundary set; if it is not a boundary set, it is not nondense; and if it is closed and not nondense, it is not a boundary set. If a set is a non-boundary set, it is dense; if it is dense, it is not nondense; if it is open and dense, it is a non-boundary set.*

These results may be seen easily from the appended diagram, the arrow indicating the direction of implication. The proofs of the state-



ments are as follows: $X^{NB}\to X^{CB}$ from IV 3 and II 3; $X^{CB}\to X^{CND}$, from II 2 and III 2; X closed and not nondense $\to X^{CB}$ in the same manner; $X^{NB}\to X^D$, from I 2 and IV 3 since $X^0\subset X$; X open and dense $\to X^{NB}$, in a similar way; $X^D\to X^{CND}$, from I 3 and III 3.

The following results are typical of non-boundary sets.

A necessary and sufficient condition that X be a non-boundary set is that X be dense and the boundary of X be nondense.

For we have

$$X^D \to (C\overline{X} = 0) \to (C\overline{\overline{X}} = 0)$$

and

$$F(X)^{ND} \to (C[\overline{C\overline{X}} + \overline{X^0}] = 0) \to (\overline{X^0} = S).$$

Conversely,

$$(\overline{X^0} = S) \to (\overline{C\overline{X}} + \overline{X^0} = S) \to (F(X)^0 = 0) \to F(X)^{ND}.$$

The product of a countable collection of non-boundary sets is a residual set; that is, the complement of a set of the first category.

For if X is such a set, then CX is the sum of a countable collection of nondense sets by IV 5.

*Every dense G_δ is residual.**

In fact, if $X = \prod X_n$, $X_n = X_n^0$, then each X_n is a non-boundary set because

$$S = \overline{X} \subset \overline{X_n} = \overline{X_n^0}.$$

In a complete metric space the product of a countable collection of non-boundary sets is not vacuous.

This is a classic theorem of R. Baire.

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* See Kuratowski, loc. cit., p. 206, Theorem V 2. I owe this reference to a referee.

TOTALLY GEODESIC EINSTEIN SPACES*

AARON FIALKOW

1. **Introduction.** An Einstein space† E_m is defined as a Riemann space V_m whose mean curvature a is a constant at each point; that is,‡

$$R_{\alpha\beta} = -ag_{\alpha\beta} \tag{1.1}$$

where $R_{\alpha\beta}$ and $g_{\alpha\beta}$ are the Ricci and metric tensors of V_m, respectively. We suppose that the dimension m of E_m exceeds 3. For every surface is an E_2 and the only E_3's are the spaces of constant curvature. In both these cases, the discussion which parallels that given in this note is obvious and trivial. Since $m>3$, it is a well known consequence of (1.1) that a is a constant throughout the space. In this note, we discuss the properties of an E_m which admits families of totally geodesic subspaces which are also Einstein spaces. It is shown that this subject is closely related to the problems of finding (a) all Einstein spaces which may be imbedded as hypersurfaces of a space of constant curvature (b) Einstein spaces which are conformal to Einstein spaces. In a restricted sense,§ we also find the first fundamental form of E_m. It is assumed that the first fundamental forms of E_m and of its subspaces which are discussed below are nonsingular although they may be indefinite.

2. **Separable Einstein spaces.** It has been shown by Bompiani‖ that the necessary and sufficient condition that the subspaces $x^p = \text{const.}$ and the orthogonal subspaces $x^i = \text{const.}$ be totally geodesic in V_m is that

$$g_{ij} = f_{ij}(x^k), \qquad g_{pq} = h_{pq}(x^r), \qquad g_{ip} = 0. \tag{2.1}$$

When the first fundamental form of V_m satisfies (2.1), it is called

* Presented to the Society, September 6, 1938.

† We represent an m-dimensional Riemann space and Einstein space by V_m and E_m, respectively.

‡ Throughout this note, $\alpha, \beta, \gamma, \delta$; h, i, j, k; p, q, r; λ, μ, ν have the ranges $1, 2, \cdots, m$; $1, 2, \cdots, n$; $n+1, n+2, \cdots, m$; $1, 2, \cdots, n-1$, respectively. An index which appears twice in an expression is to be summed over the appropriate range. A free index of a tensor equation assumes each value of its range.

§ The first fundamental form of E_m is obtained in a preferred coordinate system and depends upon the unknown first fundamental form of an arbitrary Einstein space.

‖ E. Bompiani, *Spazi Riemanniani luoghi di varietà totalmente geodetiche*, Rendiconti del Circolo Matematico di Palermo, vol. 48 (1924), p. 124.

separable, and the two forms $f_{ij}dx^i dx^j$ and $h_{pq}dx^p dx^q$ are called its *components*. It is clear that the components are the first fundamental forms of the totally geodesic V_n's and V_{m-n}'s of V_m and that the V_n's (as well as the V_{m-n}'s) are isometric.

We denote the Christoffel symbols of the first and second kind by $[\alpha\beta, \gamma]$, $\{\alpha|\beta\gamma\}$ for V_m; $[\overline{ij, k}]$, $\{\overline{i|jk}\}$ for V_n, and $[\overline{pq, r}]$, $\{\overline{p|qr}\}$ for V_{m-n}. Then it follows from (2.1) that

$$[ij, k] = [\overline{ij, k}], \qquad \{i \mid jk\} = \{\overline{i \mid jk}\}, \tag{2.2}$$

$$[pq, r] = [\overline{pq, r}], \qquad \{p \mid qr\} = \{\overline{p \mid qr}\}, \tag{2.3}$$

$$[\alpha\beta, \gamma] = 0, \qquad \{\alpha \mid \beta\gamma\} = 0 \tag{2.4}$$

if α, β, γ are not all in the same range. The Ricci tensor of V_m is defined as

$$R_{\alpha\beta} = \frac{\partial^2 \log g^{1/2}}{\partial x^\alpha \partial x^\beta} - \frac{\partial}{\partial x^\gamma}\{\gamma \mid \alpha\beta\} + \{\gamma \mid \alpha\delta\}\{\delta \mid \gamma\beta\} - \{\gamma \mid \alpha\beta\}\frac{\partial \log g^{1/2}}{\partial x^\gamma} \tag{2.5}$$

where $g = |g_{\alpha\beta}|$.

We first suppose that $n > 1$ and $m - n > 1$. Then from (2.2), (2.3), (2.4), and (2.5),

$$R_{ip} = 0, \qquad R_{ij} = \overline{R}_{ij}, \qquad R_{pq} = \overline{R}_{pq} \tag{2.6}$$

where $\overline{R}_{ij}$ and $\overline{R}_{pq}$ are the Ricci tensors of V_n and V_{m-n}, respectively. If V_m is an E_m, it follows from (1.1), (2.1), and (2.6) that

$$\overline{R}_{ij} = -af_{ij}, \qquad \overline{R}_{pq} = -ah_{pq}. \tag{2.7}$$

Hence V_n and V_{m-n} are both Einstein spaces of the same mean curvature as E_m. Conversely, by reversing the proof we find that (1.1) is a consequence of (2.1) and (2.7). This proves the following theorem:

THEOREM 2.1. *Let the first fundamental form of an Einstein space of dimensionality $m > 3$ of mean curvature a be separable into components whose dimensions exceed* 1. *Then each component is the first fundamental form of an Einstein space of mean curvature a. Conversely, only the first fundamental forms of Einstein spaces are separable in this manner.*

If $m = 4$, $n = 2$, it follows, from this theorem and the observation that a two-dimensional space of constant mean curvature has constant Riemann curvature, that each component is the first fundamental form of a space of constant curvature a. This was first proved

by Kasner.* In recent papers,† we have shown that Einstein spaces which are proper hypersurfaces of any space of constant curvature are either spaces of constant curvature themselves or are separable. By a repeated application of Theorem 2.1, we may obtain an obvious generalization which applies to an E_m whose first fundamental form is separable into more than two components.

If $n>1$ and $m=n+1$, and if (2.1) is satisfied, the curves $x^i=$const. are geodesics of E_m. In this case, equations (2.6) become

$$R_{im}=0, \qquad R_{ij}=\overline{R}_{ij}, \qquad R_{mm}=0. \tag{2.8}$$

As a consequence of (1.1), (2.1), and (2.8), we find that $\overline{R}_{ij}=0$, $a=0$, and conversely. We have proved the following theorem:

THEOREM 2.2. *A V_{n+1} which admits ∞^1 parallel totally geodesic E_n's is an E_{n+1} if and only if the mean curvature of the E_n's is zero. In this case, the mean curvature of E_{n+1} is also zero.*

3. **E_{n+1} with totally geodesic E_n's.** We suppose $m=n+1$ and that E_{n+1} admits ∞^1 totally geodesic E_n's which are not parallel. Since the first fundamental form of the necessarily isometric E_n's is non-singular, it follows that the normals to the E_n's in E_{n+1} are not null vectors.‡ Hence, in accordance with a slightly weaker form of the theorem of Bompiani quoted above,

$$g_{ij}=f_{ij}(x^k), \qquad g_{n+1,n+1}=eH^2(x^i, y), \qquad g_{i,n+1}=0, \tag{3.1}$$

where $y=x^{n+1}$, e is $+1$ or -1, and $f_{ij}dx^idx^j$ is the first fundamental form of each E_n. Since the hypersurfaces are not parallel, it follows that $H(x^i, y)$ cannot be a function of y only but must involve the x^i. Because of (3.1), we find that (2.2) holds and (2.4) is true if one of α, β, γ is $n+1$ and the other two lie in the range $1, 2, \cdots, n$. Also

$$\left\{n+1 \middle| i\ n+1\right\}=\frac{\partial \log H}{\partial x^i}, \quad \left\{n+1 \middle| n+1\ n+1\right\}=\frac{\partial \log H}{\partial y},$$
$$\left\{i \middle| n+1\ n+1\right\}=-eH^2\frac{\partial \log H}{\partial x^j}\cdot g^{ij}, \tag{3.2}$$

* E. Kasner, *An algebraic solution of the Einstein equations*, Transactions of this Society, vol. 27 (1925), pp. 101–105, and *Separable quadratic differential forms and Einstein solutions*, Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 95–96.

† A. Fialkow, *Einstein spaces in a space of constant curvature*, Proceedings of the National Academy of Sciences, vol. 24 (1938), pp. 30–34, and *Hypersurfaces of a space of constant curvature*, Annals of Mathematics, (2), vol. 39 (1938), pp. 762–785. The term "proper" is defined in these papers.

‡ L. P. Eisenhart, *Riemannian Geometry*, 1926, p. 144.

where g^{ij} are the contravariant components of g_{ij}. From (2.2), (2.4), (2.5), and (3.2),

$$R_{n+1,n+1} = -\frac{\partial}{\partial x^i}\left\{i \middle| n+1\ n+1\right\} + \left\{n+1 \middle| n+1\ i\right\}\left\{i \middle| n+1\ n+1\right\} - \left\{i \middle| n+1\ n+1\right\}\frac{\partial \log f^{1/2}}{\partial x^i}, \tag{3.3}$$

$$R_{i,n+1} = 0, \tag{3.4}$$

$$R_{ij} = \overline{R}_{ij} + (\log H)_{,ij} + (\log H)_{,i}(\log H)_{,j}, \tag{3.5}$$

where the comma denotes covariant differentiation with respect to the form $f_{ij}dx^i dx^j$, and $f = |f_{ij}|$. Since E_{n+1} and E_n are Einstein spaces, (1.1) and

$$\overline{R}_{ij} = -bf_{ij} \tag{3.6}$$

are true.

It follows from (1.1), (3.1), and (3.6) that (3.5) becomes

$$H_{,ij} = cHf_{ij} \tag{3.7}$$

where

$$c = b - a. \tag{3.8}$$

The integrability conditions of (3.7) are

$$H_{,ijk} - H_{,ikj} = H_{,h}\overline{R}^h_{ijk}$$

which become, by virtue of (3.7),

$$H_{,h}\overline{R}^h_{ijk} = c(H_{,k}f_{ij} - H_{,j}f_{ik}).$$

The tensor $\overline{R}^h_{ijk}$ is the Riemann curvature tensor of E_n. If we multiply this equation by the contravariant components f^{ij} of the metric tensor of E_n and sum for i, j, we find after using (3.6) that

$$[c(n-1) + b]H_{,k} = 0.$$

Since $H_{,k}$ cannot be zero for every value of k, the last equation and (3.8) show that

$$nb = (n-1)a \tag{3.9}$$

is a necessary condition that (3.7) have a solution.

Since $\partial \log (f)^{1/2}/\partial x^i = \left\{\overline{k} \middle| \overline{ki}\right\}$, after using (3.2), equation (3.3) becomes

$$R_{n+1,n+1} = eH\left[\frac{\partial}{\partial x^i}(f^{ij}H_{,j} + f^{ij}H_{,j}\{k\,|\,i\}\right]$$

or

$$R_{n+1,n+1} = eH\Delta_2 H \tag{3.10}$$

where $\Delta_2 H = f^{ij}H_{,ij}$. From (3.7), (3.8), (3.9), and (3.10),

$$R_{n+1,n+1} = -aeH^2.$$

According to (3.1), this equation and (3.4) both obey (1.1). Hence E_{n+1} will admit ∞^1 nonparallel totally geodesic E_n's if and only if a solution of (3.7) exists.

Since $H(x^i, y)$ is not independent of the x^i, we may choose coordinates x^i so that $H = x^n$ for some fixed value of y and such that $f_{n\lambda} = 0$. Then (3.7) becomes

$$-\{\overline{n\,|ij}\} = cx^n f_{ij}.$$

Now Brinkmann* has shown that E_n admits a solution of these equations if and only if its metric tensor satisfies

$$f_{nn} = (cx^{n^2} + d)^{-1},$$
$$f_{\lambda\mu} = (cx^{n^2} + d)F_{\lambda\mu}(x^\nu), \qquad f_{n\lambda} = 0, \qquad d \text{ constant}, \tag{3.11}$$

and the form $F_{\lambda\mu}(x^\nu)dx^\lambda dx^\mu$ is the first fundamental form of an E_{n-1}. According to Brinkmann, (3.11) is the necessary and sufficient condition that E_n be conformal to another Einstein space by means of a transformation $d\bar{s} = \sigma ds$ with $\Delta_1\sigma \neq 0$ where $\Delta_1\sigma = f^{ij}\sigma_{,i}\sigma_{,j}$. If we suppose $H = x^n$ for all values of y, using (3.1) we may write the first fundamental form of one E_{n+1} which satisfies the conditions of the problem as

$$ds^2 = f_{ij}dx^i dx^j + ex^{n^2}dx^{n+1^2} \tag{3.12}$$

where the f_{ij} satisfy (3.11). This proves the following theorem:

THEOREM 3.1. *A one-parameter family of isometric E_n's may be imbedded as ∞^1 nonparallel totally geodesic hypersurfaces of an E_{n+1} if and only if each E_n may be mapped conformally on another Einstein space by means of a function σ with $\Delta_1\sigma \neq 0$. If a and b are the mean curvatures of E_{n+1} and E_n, respectively, then $nb = (n-1)a$.*

We now briefly consider the conditions under which the E_n de-

* H. W. Brinkmann, *Einstein spaces which are mapped conformally on each other*, Mathematische Annalen, vol. 94 (1925), pp. 123–125.

fined by (3.11) admits solutions $H(x^i, y)$ of the equations (3.7) other than $H=x^n$. By methods similar to those hitherto employed, we find that the most general solution for H of the form $H=H(x^n, y)$ is

$$H = \alpha(y)\cdot x^n + \beta(y), \qquad c = 0, \tag{3.13}$$

or

$$H = \alpha(y)\cdot x^n, \qquad c \neq 0, \tag{3.14}$$

where $\alpha(y)$ and $\beta(y)$ are arbitrary functions of y. We note that the E_{n+1} obtained by using the H defined by (3.14) coincides with (3.12).

It can be shown that solutions for H which involve some of the x^r do not exist unless the E_n defined by (3.11) may be mapped conformally on another Einstein space in more than one way. Hence, if this is not the case, the E_n's may only be imbedded in the unique E_{n+1} defined by (3.12) if $c\neq 0$ and only in the E_{n+1}'s defined by (3.1), (3.11), and (3.13) if $c=0$. In this last case, $a=b=0$.

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CONCERNING THE BOUNDARY OF A COMPLEMENTARY DOMAIN OF A CONTINUOUS CURVE*

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Much study by various investigators has been given to the nature of the boundary of a complementary domain of a locally compact continuous curve in the plane and in certain other spaces.† It is the purpose of this paper to continue this investigation in less restricted spaces which satisfy the Jordan curve theorem and to establish certain results (from which many of the known results follow immediately) in such a way as to bring out what is essential for their validity.

It is first necessary to establish the following lemma.

LEMMA A. *If a locally compact nondegenerate continuous curve M in a complete Moore space contains no simple triod, then M is a simple continuous curve.*‡

* Presented to the Society, December 30, 1938.

† See the bibliography and Chapter 4 of R. L. Moore's *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, New York, 1932. Hereinafter, this book will be referred to as *Foundations*, and the reader is referred to it for many theorems and the definition of certain terms and phrases used in this paper.

‡ A *complete Moore space* is a space satisfying Axioms 0 and 1 of *Foundations*. A *simple continuous curve* is either a simple continuous arc, a simple closed curve, an open curve, or a ray.

PROOF. With the help of Theorems 118 and 120 in Chapter 1 and the arguments for Theorems 6 and 7 in Chapter 2 of *Foundations*, it is easy to see that if "point" is interpreted to mean a point of M and "region" is interpreted to mean an open subset of M, then as a space, M satisfies Axioms 0–2 of *Foundations* and "limit point" is invariant. Suppose that H is any compact subcontinuum of M. By Theorem 65 in Chapter 1 of *Foundations*, H contains two distinct points, A_1 and A_2, which are non-cut points of H. Suppose that P is a point of H distinct from A_1 and A_2 which is a boundary point of H with respect to M. Let D denote a connected open subset of M containing P such that $\overline{D}$ contains neither A_1 nor A_2. The set D contains a point A_3 of $M-H$. Let d_1 and d_2 denote mutually exclusive connected open subsets of M containing A_1 and A_2, respectively, but containing no point of D. For each point X of $H-(A_1+A_2)$, there exists a connected open subset d_X of M containing X but not containing A_1, A_2, or A_3. Let d denote $\sum d_X$ for all points X of $H-(A_1+A_2)$. Then since $H-A_2$ is connected, $D+d_1+d$ is a connected open subset of $M-A_2$ containing A_1 and A_3. By Theorem 1 in Chapter 2 of *Foundations*, there exists an arc T_1 in $D+d_1+d$ from A_1 to A_3. Since $H-A_1$ is connected, d_2+d is a connected open subset of M containing $H-A_1$ and a point of T_1. Let T_2 denote an arc in d_2+d irreducible from A_2 to T_1. Then T_1+T_2 is a simple triod contrary to hypothesis. Hence no point of H distinct from A_1 and A_2 is a boundary of H with respect to M, and consequently no compact subcontinuum of M has more than two boundary points with respect to M. Then by Theorem 20′ in Chapter 2 of *Foundations*, M is a simple continuous curve.

COROLLARY. *If a compact nondegenerate continuous curve in a complete Moore space contains no simple triod, then it is either an arc or a simple closed curve.*†

The results of this paper that follow assume that Axioms 0–4 of *Foundations* hold true. Let S denote the set of all points.

THEOREM 1. *Suppose that K is a locally compact continuum lying in the boundary of a connected domain D. Then in order that K be a continuous curve it is necessary and sufficient that K be a subset of a continuous curve M which contains no point of D.*

PROOF. The necessity is obvious. It remains only to prove the sufficiency. Suppose, on the contrary, that K is not connected im kleinen at a point O of K. Then there exists a domain Q_1 containing O such

† Cf. Theorem 71 in Chapter 4 of *Foundations*.

that $Q_1 \cdot K$ is compact and the component of $Q_1 \cdot K$ which contains O is not open with respect to K at O. Let Q_2 denote a domain containing O and lying together with its boundary in Q_1. There exist two sequences of points $O_1, O_2, O_3, \cdots$ and $P_1, P_2, P_3, \cdots$ such that (1) for each n, $(n=1, 2, 3, \cdots)$, O_n and P_n belong to the same component of $Q_2 \cdot K$, (2) if $m \neq n$, then O_m and O_n belong to different components of $Q_1 \cdot K$, and (3) $O_1, O_2, O_3, \cdots$ converges to O and $P_1, P_2, P_3, \cdots$ converges to a point P of $Q_1 \cdot K$ distinct from O. Let U and V denote two connected open subsets of M containing O and P, respectively, such that $\overline{U} \cdot \overline{V} = 0$. By (3) there exist three integers n_1, n_2, and n_3 such that O_{n_1}, O_{n_2}, O_{n_3} and P_{n_1}, P_{n_2}, P_{n_3} lie in U and V, respectively. For each i, $(i=1, 2, 3)$, let T_i denote the component of $Q_2 \cdot K$ that contains $O_{n_i} + P_{n_i}$. By (2), $\overline{T}_i \cdot \overline{T}_j = 0$ if $i \neq j$. Hence, $\overline{T}_1$, $\overline{T}_2$, and $\overline{T}_3$ are mutually exclusive compact continua lying in $Q_1 \cdot K$, and each contains both a point of U and a point of V. For each i, $(i=1, 2, 3)$, there exists a finite collection H_i of sets such that (a) each element of H_i is a connected open subset of M containing a point of $\overline{T}_i$, (b) H_i covers $\overline{T}_i$, (c) no element of H_i contains a point of an element of H_j if $i \neq j$, and (d) no element of H_i contains both a point of U and a point of an element of H_i containing a point of V. The set H_i^* is a connected domain with respect to M containing $\overline{T}_i$.† By Theorem 10 in Chapter 2 of *Foundations*, there exists, for each i, $(i=1, 2, 3)$, an arc Y_iZ_i in H_i^* from a point Y_i of U to a point Z_i of V. Let Y_1Y_2 and Y_1Y_3 denote arcs lying in U and Z_1Z_2 and Z_1Z_3 denote arcs lying in V, with end points as indicated. Then there exist three arcs AW_1B, AW_2B, and AW_3B from a point A of U to a point B of V lying in $Y_1Y_2 + Y_1Z_1 + Z_1Z_2$, $Y_1Y_2 + Y_2Z_2 + Z_1Z_2$, and $Y_1Y_3 + Y_3Z_3 + Z_1Z_3$, respectively, which have only their end points in common. Let ω, "the point at infinity," be a point of D. From Theorems 4 and 5 in Chapter 3 of *Foundations*, it follows that the sum of one pair of these arcs, say $AW_1B + AW_3B$, forms a simple closed curve J whose interior contains the other arc less its end points, that is, $AW_2B - (A+B)$. By (d) above, it is clear that some element d of H_2 contains a point of AW_2B and a point X of T_2 but no point of $U+V$. By (c), d contains no point of $H_1^* + H_3^*$. Consequently, d contains no point of J since J lies in $U + H_1^* + V + H_3^*$. But X is a boundary point of D. Hence $D+d$ is a connected set containing no point of J but containing ω and a point of AW_2B in the interior of J, which is a contradiction.

EXAMPLE. *Theorem* 1 *is false if the stipulation that K be locally com-*

† H_i^* denotes the sum of the elements of H_i.

pact is omitted. This may be seen in an example discovered by R. L. Moore some years ago but as yet unpublished. This example may be roughly described as follows. In a euclidean 3-space for each n, $(n=1, 2, 3, \cdots)$, let U_n denote a circular cylinder whose radius is

$$\frac{1}{2}\left[\frac{1}{n}-\frac{1}{n+1}\right]$$

and whose axis is a line parallel to the z axis passing through $(1/n, 0, 0)$. Let S denote the set of all points P such that either (1) P is in the xy-plane but is not, for any n, $(n=1, 2, 3, \cdots)$, within U_n, or (2) P is in the plane $z=1$ and is, for some integer n, within U_n, or (3) P is, for some integer n, in U_n and either in or between the planes $z=0$ and $z=1$. If "limit point" is given the ordinary 3-dimensional sense, S satisfies Axioms 0-5 of *Foundations*. Let K denote the intersection of the xz-plane with S; let M denote all of the points of S either on or between the two planes $y=0$ and $y=1$; and let D denote the component of $S-M$ which contains $(0, -1, 0)$. Then M is a continuous curve in S and K is a continuum in S lying both in M and in the boundary of M. But obviously K is not connected im kleinen at $(0, 0, 0)$.

This example should be remembered in connection with certain results to follow—Theorem 8, in particular.

Theorem 1 establishes the truth of the following two theorems.

THEOREM 2. *Every component of the boundary of a complementary domain of a locally compact continuous curve is a locally compact continuous curve.*

THEOREM 3. *If the boundary of a complementary domain of a locally compact continuous curve is connected, it is itself a locally compact continuous curve.*†

THEOREM 4. *If D and Q are two mutually exclusive connected domains whose boundaries contain a simple closed curve J, then J separates D from Q.*

PROOF. Suppose, on the contrary, that D and Q both lie in I, one of the complementary domains of J. Let ω, "the point at infinity," be a point in the other complementary domain of J. Then I is the interior of J. There exists in D an arc segment T whose end points, A and B, lie on J.‡ There exist two points, C and F, of J which are

† Cf. Theorem 40 in Chapter 4 of *Foundations*.

‡ A connected open subset of non-end points of a simple continuous curve is called a *segment*. An *arc segment* is a segment of an arc.

separated on J by A and B. Let I_1 and I_2 denote the interiors of ACB (of J)$+T$ and AFB (of J)$+T$, respectively. By Theorem 4 in Chapter 3 of *Foundations*, $I=T+I_1+I_2$. Since Q lies in I and contains no point of T, Q is a subset either of I_1 or of I_2. If Q lies in I_1, then F is not in the boundary of Q, and if Q lies in I_2, then C is not in the boundary of Q. In either case, some point of J is not in the boundary of Q which is a contradiction.

THEOREM 5. *If D is a connected domain and E is a point of $S-\overline{D}$, then the outer boundary of D with respect to E is either acyclic or a simple closed curve.*†

PROOF. Suppose that the boundary of Q, the component of $S-\overline{D}$ which contains E, contains a simple closed curve J. Then J is in the boundary of both D and Q. If the boundary of Q contains a point P not in J, then P is obviously in the boundary of D and $D+P+Q$ is a connected subset of $S-J$. But this contradicts Theorem 4. Hence J is the complete boundary of Q.

THEOREM 6. *If D is a complementary domain of a locally compact continuous curve M, and E is a point of $S-\overline{D}$, then every component of the outer boundary of D with respect to E is a continuous curve.*

PROOF. Let C denote a component of the boundary of Q, the complementary domain of $\overline{D}$ which contains E. Then C is a subset of a component K of the boundary of D. By Theorem 2, K is a locally compact continuous curve. But K contains no point of Q; hence, by Theorem 1, C is a continuous curve.

THEOREM 7. *If D is a complementary domain of a locally compact continuous curve M, and E is a point of $S-\overline{D}$, then every component of the outer boundary of D with respect to E is atriodic.*

PROOF. Let C denote a component of the boundary of Q, the complementary domain of $\overline{D}$ which contains E. Suppose that C contains a triod. Using the preceding theorem and Theorem 10 in Chapter 2 of *Foundations*, it can be shown that C contains three arcs A_1O, A_2O, and A_3O which have only the point O in common. Let d_1, d_2, and d_3 denote three mutually exclusive connected open subsets of M containing A_1, A_2, and A_3, respectively, but not containing O. With the help of Theorems 1, 2, and 10 of Chapter 2 of *Foundations*, it is easy to see that, for each i, $(1\leqq i\leqq 3)$, there exists in $Q+d_i$ an arc PA_i' from a point P of Q to a point A_i' of A_iO such that P is the only point

† Cf. Theorem 41 in Chapter 4 of *Foundations*.

that PA_1', PA_2', and PA_3' have in common. Let ω, "the point at infinity," be a point of D, and for each i, $(1 \leqq i \leqq 3)$, let $PA_i'O$ denote PA_i' plus the interval of A_iO from A_i' to O. It follows from Theorems 4 and 5 in Chapter 3 of *Foundations* that the sum of one pair of these arcs, say $PA_1'O+PA_3'O$, is a simple closed curve J whose interior I contains the internal points of the other arc, $PA_2'O$. Since J contains no point of D, I contains no point of D. But I contains a boundary point A_2' of D, which is a contradiction.

THEOREM 8. *If D is a complementary domain of a locally compact continuous curve M, and E is a point of $S-\overline{D}$, then every nondegenerate component of the outer boundary of D with respect to E is a simple continuous curve.*†

Theorem 8 follows from Lemma A and Theorems 6 and 7.

THEOREM 9. *If* (1) *D is a complementary domain of a locally compact continuous curve M,* (2) *E is a point of $S-\overline{D}$,* (3) *C is a component of β, the outer boundary of D with respect to E, and* (4) *X is a non-end point of C, then X is not a limit point of $\beta-C$.*

PROOF. Let Q denote the component of $S-\overline{D}$ which contains E; and suppose that the theorem is false. Then there exists an arc A_1XA_2 which contains X as an internal point and a connected open subset d_X of M containing X but containing neither A nor B. The set d_X contains a point A_3 of $\beta-C$. Let A_3O denote an arc in d_X irreducible from A_3 to a point O of A_1XA_2, and let A_1O and A_2O denote the intervals of A_1XA_2 from A_1 and A_2, respectively, to O. Let d_1, d_2, and d_3 denote three mutually exclusive connected open subsets of M containing A_1, A_2, and A_3, respectively, but not containing O. With the help of Theorems 1, 2, and 10 in Chapter 2 of *Foundations*, it is easy to see that for each i, $(1 \leqq i \leqq 3)$, there exists an arc PA_i' lying in $Q+d_i$ which is irreducible from a point P of Q to A_iO such that P is the only point that PA_1', PA_2', and PA_3' have in common. For each i, $(1 \leqq i \leqq 3)$, let $PA_i'O$ denote PA_i' plus the interval of A_iO from A_i' to O, and let A_iA_i' denote the interval of A_iO from A_i to A_i'. Let ω, "the point at infinity," be a point of D. It follows from Theorems 4 and 5 in Chapter 3 of *Foundations* that the sum of one pair, say $PA_1'O+PA_3'O$, of the arcs $PA_1'O$, $PA_2'O$, and $PA_3'O$ is a simple closed curve J whose interior I contains the internal points of the other arc, $PA_2'O$. Since J contains no point of D, I contains no point of D. But A_2A_2' contains an internal point A_2' of $PA_2'O$ and

† Cf. Theorem 41 in Chapter 4 of *Foundations*.

contains no point of J. Hence I contains A_2, a boundary point of D, which is a contradiction.

THEOREM 10. *If D is a complementary domain of a compact continuous curve and E is a point of $S-\overline{D}$, then the outer boundary of D with respect to E is either a simple closed curve or the sum of the elements of a countable collection G of mutually exclusive arcs and a totally disconnected closed point set H such that $G^* \cdot H$ is the set of all end points of the arcs of G.*

Theorem 10 follows from Theorems 5, 8, and 9.

THEOREM 11. *If (1) D is a complementary domain of a locally compact continuous curve M, (2) E is a point of $S-\overline{D}$, (3) C is a component of β, the outer boundary of D with respect to E, and (4) β and the boundary of D are identical, then D plus the non-end points of C is a connected, connected im kleinen, inner limiting set.*†

PROOF. Let H denote D plus the non-end points of C. Obviously, H is a connected inner limiting set and is connected im kleinen at all of the points of D. Suppose that X is a non-end point of C. Then from Theorems 8 and 9 it is easy to see that there exists a region R which contains X but contains neither an end point of C nor any point of $\beta-C$. Let T denote the component of $R \cdot C$ which contains X, and let R_1 denote the component of $R-R \cdot (C-T)$ which contains C. Obviously T is a segment, R_1 is a domain, and since any point of $R_1 \cdot H$ may be joined to T by an arc in R_1, $R_1 \cdot H$ is a connected subset of R which is open with respect to H. Since this is true for any region R containing X which contains neither an end point of C nor any point of $\beta-C$, H is connected im kleinen at X.

THEOREM 12. *If K is the boundary of a complementary domain D of a locally compact continuous curve M, β is the outer boundary of D with respect to a point E of $S-\overline{D}$, and H is a component of $K-\beta$, then $\overline{H}$ is a continuous curve having only one point in β.*‡

PROOF. Let Q denote the component of $S-\overline{D}$ which contains E. Then β is the boundary of Q. Evidently $\overline{H}$ contains at least one point of β. Suppose that $\overline{H}$ contains two points, A_1 and A_2, of β. Let d_1 and d_2 denote two mutually exclusive connected open subsets of M containing A_1 and A_2, respectively. By Theorems 2 and 10 in Chapter 2 of *Foundations*, $H+d_1+d_2$ contains an arc T_H from A_1 to A_2.

† Cf. Theorem 18 in Chapter 3 of *Foundations*.

‡ Cf. Theorem 43 in Chapter 4 of *Foundations*.

By Theorems 1 and 2 in Chapter 2 of *Foundations*, it follows that $Q+d_1+d_2$ contains an arc T_Q irreducible from $T_H \cdot d_1$ to $T_H \cdot d_2$. Then T_H+T_Q contains a simple closed curve J which contains a point X of H and a point Y of Q but which contains no point of D. Let I denote the complementary domain of J which contains no point of D. The domain I contains no point of $\overline{D}$ and consequently no point of β. Hence $I+X+Y$ is a connected point set lying in Q since it contains a point of Q but no point of β. But since X is a point of $\overline{D}$, this is a contradiction. Consequently $\overline{H}$ has only one point O in β. Furthermore, it is easy to see that $\overline{H}$ is connected im kleinen; for it is evident that $\overline{H}$ is connected im kleinen at all of its points except possibly O, and if d is any connected open subset of M containing O, the components of $d-O$ which contain points of H together with O form a connected open subset of $\overline{H}$.

However, despite Theorems 8, 9, and 12, Theorem 11 is false if condition (4) is omitted. Speaking roughly, condition (4) may be omitted and the theorem remain true if S does not contain both "hills" and "holes."

The reader should note that Theorems 6 to 12 inclusive remain true if, instead of postulating that D is a complementary domain of a locally compact continuous curve, it is postulated that D is a complementary domain of a continuous curve and the boundary of D is locally compact. This is quite evident since the property of local compactness is not used in any proof other than the proof of Theorem 1. Of course, the boundaries of the domains involved must be locally compact in order to make the use of Theorem 1 valid.

THE UNIVERSITY OF TEXAS

ON THE SINGULAR POINT LOCUS IN THE THEORY OF FIELDS OF PARALLEL VECTORS

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The surface of an ordinary (circular) half cone is frequently employed as an illustration of a surface on which parallel displacement of a vector around a closed circuit may produce a change in the vector. It may appear at first sight that the reason for this is related to the fact that the circuit in question encircles the vertex of the cone and the vertex is in some sense a singular or exceptional point. But this cannot be correct since we can remove the vertex from the surface under consideration without effecting the parallel displacement of the vector. If it is then argued that the resulting surface is not simply connected and that the above phenomenon depends on the fact that the circuit along which the vector is displaced cannot be shrunk to a point, we can offset this by smoothing out the surface in the neighborhood of the vertex so that it becomes a continuous, differentiable, and simply connected surface (which is evidently possible). Clearly then, such considerations do not provide us with the inner reason as to why the parallel displacement of a vector around certain closed circuits on the surface yields the original vector while for other circuits a different vector is obtained as result of the parallel displacement.

A satisfactory answer to the above question is contained in a general theory of the parallel displacement of vectors in an affinely connected space by Mayer and Thomas, *Fields of parallel vectors in non-analytic manifolds in the large*, Compositio Mathematica, vol. 5 (1938), pp. 193–207. It can be shown that an affinely connected space of class C^r, where $r \geqq n+1$ and n is the dimensionality of the space, breaks up *in virtue of its intrinsic nature* into a finite or infinite number of open point sets K, called components, with each of which there is associated a definitely determined integer m having a value from zero to n inclusive. Denote by K_m any component K for which m is the associated integer. Then any open, connected, and simply connected point set $O \subset K_m$ admits exactly m independent fields of parallel vectors. The definition of the components K is as follows: Consider the set of equations

$$(E_0)\xi^\mu B^\alpha_{\mu\beta\gamma} = 0;\ (E_1)\xi^\mu B^\alpha_{\mu\beta\gamma,\delta_1} = 0;\ \cdots;\ (E_n)\xi^\mu B^\alpha_{\mu\beta\gamma,\delta_1,\cdots,\delta_n} = 0$$

as equations for the determination of the n quantities ξ^μ, the co-

efficients B being the components of the curvature tensor and its successive covariant derivatives. We shall say that a point P of the space is a *regular point with respect to the system* $(E_0)+\cdots+(E_t)$ if there exists a neighborhood $U(P)$ in which the rank of the matrix of the coefficients B of this system is constant. Otherwise the point P will be said to be *singular with respect to the above system*. It is easily demonstrated that the set of singular points so defined is nowhere dense in the space. Denote by R_{n-1} the set of all points regular with respect to the system $(E_0)+\cdots+(E_{n-1})$. A component $K=K(P)$ is defined as the greatest open connected point set in R_{n-1} which contains P. Evidently $K(P)=K(Q)$ if $Q\subset K(P)$. It can be shown that the matrix of the system $(E_0)+\cdots+(E_{n-1})$ has a constant rank in any component $K(P)$. If the rank of this matrix is ρ in $K(P)$, then this component $K(P)$ is one of the above components K_m where $m=n-\rho$. It is furthermore proved in the above paper that if Q is any point in the above point set $O\subset K_m$ and if ξ_Q is any solution vector of the system $(E_0)+\cdots+(E_{n-1})$ at Q, then a field of parallel vectors is generated in O by the process of parallel displacement of the vector ξ_Q to the various points of the set O. Taking m independent solution vectors $\xi_Q^1,\cdots,\xi_Q^m$ at Q, we thus obtain the existence of m independent fields of parallel vectors in O as above stated.

In the case of the cone with vertex deleted, we have a (locally) flat affine space which falls under the above general theory. This space contains only one component K_n, so that there exist n independent fields of parallel vectors in any of the open point sets O; hence if any vector is displaced by parallel displacement around a circuit lying entirely in O, it will return to the original vector. If, on the other hand, the circuit is not contained in a simply connected region O, the original vector need not be obtained as a result of the parallel displacement. If we smooth out the cone at the vertex so as to obtain a simply connected space of class C^r, this space must contain in addidition to the component K_n at least one component K_m where $m\neq n$. Otherwise, we would have a contradiction with the fact that there exist circuits about which the parallel displacement of a vector does not yield the original vector.

When the affinely connected space is analytic, the components K into which the space breaks up in accordance with the above general theory lose their real significance. Indeed, the entire space in the analytic case appears to play the same role as the various components K under the non-analytic hypothesis. See Thomas, *Fields of parallel vectors in the large*, Compositio Mathematica, vol. 3 (1936),

pp. 453–468. The fact that the various components K which may exist in an analytic affinely connected space are without especial significance in the theory of the parallel displacement of vectors in this space is an objection to the above definition of these components. It is the primary purpose of the following paper to modify the definition of the components K of an affinely connected space so as to eliminate all nonessential components at least as far as this is possible in view of the continuity and differentiability properties which we have at our disposal. As a consequence of this new definition, an analytic affinely connected space will be without singular points and hence (if topologically connected) will contain only one component which is in strict conformity with the above mentioned character of the analytic space.

Let S be an affinely connected space of class C^u. If u is a finite integer, this means that the components of the affine (or linear) connection L are continuous and have all continuous derivatives to the order u inclusive; if $u=\infty$, the components of the affine connection are continuous and have all continuous partial derivatives without exception; and if $u=\omega$, these components are analytic functions of the coordinates of the various admissible coordinate neighborhoods by which the space is covered.

If u is a finite integer, we put $u=r+2$ and assume $r\geqq n-1$, where n is the dimensionality of the space, so that we can construct the sets of equations (E_0), (E_1), $\cdots$, (E_r), (E_{r+1}) which correspond to and ininclude those above considered. In the case of a space of class C^∞ or C^ω (analytic affinely connected space) the infinite sequence of equations (E_0), (E_1), (E_2), $\cdots$ can be constructed over the space. Denote by M the matrix of the system $(E_0)+\cdots+(E_r)$ for the space of class C^{r+2} and the (infinite) matrix of the system $(E_0)+(E_1)+(E_2)+\cdots$ in the case of an analytic space or space of class C^∞.

We shall say that a point P of S is *regular* if there exists a neighborhood $U(P)$ in which the rank of the above matrix M is constant. Otherwise P will be said to be a *singular* point. Denote by R the set of all regular points in S. By a *component* $K(P)$ we shall mean the greatest open connected point set in R which contains the point P. Then $K(P)=K(Q)$ if $Q\subset K(P)$. It can be shown that if P is any regular point of a space S of class C^u for which $u=r+2$ is finite, then the system (E_{r+1}) can be expressed linearly and homogeneously in terms of the systems (E_0), $\cdots$, (E_r) in some neighborhood $U(P)$, the coefficients in these expressions being continuous functions of the coordinates of $U(P)$. When S is an analytic space or space of class

C^*, it is immediately evident that there exists a neighborhood $U(P)$ of any regular point P in which the system (E_t), for some value of the integer t, can be expressed linearly and homogeneously in terms of the systems $(E_0), \cdots, (E_{t-1})$ with coefficients which are analytic or functions of class C^*, respectively, of the coordinates of $U(P)$. On the basis of the procedures which are employed in the paper by Mayer and Thomas (loc. cit.), the following two theorems can be established.

THEOREM I. *The rank of the matrix M is contant in any component of the space S.*

THEOREM II. *If ρ is the rank of the matrix M in a component K of an affinely connected space S of class C^u and if O is any open, connected, and simply connected point set in K, then there are exactly $n-\rho$ independent fields of parallel vectors in O and these fields can be generated by the parallel displacement to the various points of O of the $n-\rho$ independent solution vectors, at any point $Q \subset O$, of the system $(E_0)+\cdots+(E_r)$ if $u=r+2$ and $r \geqq n-1$ is a finite integer where n is the dimensionality of the space S, or of the infinite system $(E_0)+(E_1)+\cdots$ if the space S is analytic or of class C^*.*

It is easily seen that any component on the basis of the previous definition is contained entirely in one of the components which enter in the above theorems. Also points which were singular, as previously defined, may become regular points as a result of the modified definition; and in consequence of this, two or more of the components (as previously defined) may merge into a single component. We shall show that in the case of an analytic space this merging is complete; that is, there exists only a single component K in any (topologically connected) analytic affinely connected space.

Let P be any point of an analytic space S at which the matrix M has rank $\rho<n$. Let ξ be any solution vector of the system $(E_0)+(E_1)+\cdots$ at P. Let $C(t)$, where $0 \leqq t \leqq 1$, be an analytic curve joining the point P to any other point Q such that $P=C(0)$ and $Q=C(1)$, which is always possible if S is topologically connected. See Thomas, *Arcs in affinely connected spaces*, Annals of Mathematics, (2), vol. 38 (1937), pp. 120–130. Displace the vector ξ at P by parallel displacement along $C(t)$ to obtain an analytic vector $\xi(t)$ defined along the curve C. Along C we may then define the analytic following functions

$$F^{\alpha}_{\beta\gamma}(t) = \xi^{\mu}(t)B^{\alpha}_{\mu\beta\gamma}, \quad F^{\alpha}_{\beta\gamma\delta}(t) = \xi^{\mu}(t)B^{\alpha}_{\mu\beta\gamma,\delta}, \cdots$$

all of which vanish at P, that is, at $t=0$, by the above hypothesis. It follows immediately that the successive *absolute derivatives* of the tensors F all vanish at P; that is,

$$\frac{D}{Dt}\cdots\frac{D}{Dt}F^{\alpha}_{\beta\gamma}(0)=0,\quad \frac{D}{Dt}\cdots\frac{D}{Dt}F^{\alpha}_{\beta\gamma\delta}(0)=0,\ \cdots.$$

From the first set of these equations it can easily be shown that all the *absolute extensions* of the first tensor F vanish at P, from the second set of these equations all the absolute extensions of the second tensor F vanish at P, and so on; that is,

$$\frac{D^{m}F^{\alpha}_{\beta\gamma}(0)}{Dt^{m}}=0,\quad \frac{D^{m}F^{\alpha}_{\beta\gamma\delta}(0)}{Dt^{m}}=0,\ \cdots,\qquad m=1,2,3,\cdots.$$

The definition of the absolute derivative is, of course, well known, but the definition of the mth absolute extension is perhaps not well known. Let us note, therefore, that the above mth absolute extensions are defined in the following manner: Introduce normal coordinates y^{α} in a neighborhood of the point P. Denote, for example, the components of the first of the above tensors F by $f^{\alpha}_{\beta\gamma}(t)$ with respect to the normal coordinate system. Then

$$\left[\frac{D^{m}F^{\alpha}_{\beta\gamma}}{Dt^{m}}\right]_{P}=\left[\frac{d^{m}f^{\alpha}_{\beta\gamma}}{dt^{m}}\right]_{P},$$

where the right-hand member contains the ordinary mth derivative of the components $f^{\alpha}_{\beta\gamma}(t)$, and a similar definition applies in the case of any tensor. It follows, therefore, that the successive coefficients in the power series expansions of the components of each of the tensors F vanish at $t=0$; and since these components are analytic functions of t along the entire curve $C(t)$, they must vanish along this curve; that is, in the interval $0\leqq t\leqq 1$. In particular, we have $F^{\alpha}_{\beta\gamma}(1)=0$, $F^{\alpha}_{\beta\gamma\delta}(1)=0,\cdots$; that is, by parallel displacement of the solution vector ξ of the system $(E_0)+(E_1)+\cdots$ at P along the analytic arc C joining P to Q, we obtain a vector ξ' at Q which is also a solution vector of this system. Since independent solution vectors of the above system at P (or Q) remain independent under parallel displacement along the curve C joining P and Q, it follows that the matrix M has the same rank at P as it has at the point Q. In other words, the space S contains no singular points.

THEOREM III. *An analytic affinely connected space S can contain no singular points.*

THEOREM IV. *An analytic affinely connected space S which is topologically connected contains only a single component K.*

Let M' be the matrix of the system $(E_0)+\cdots+(E_{n-1})$, and denote by $R_m=0$, where $m=1,\cdots,n$, the equations obtained by equating to zero all determinants of order $n-m+1$ in the matrix M'. Let S be an affinely connected space of class C^u, and suppose that the conditions $R_m=0$ are satisfied over S, where m has one of the above values $1,\cdots,n$. Let K be any component of S. Since the set of points in S which are singular with respect to the system $(E_0)+\cdots+(E_{n-1})$ is nowhere dense, it follows that there is a point $P \subset K$ which is regular with respect to this system. Then there exists a neighborhood $U(P)$ in which M' has a constant rank ρ, such that the (E_n) are expressible linearly and homogeneously in terms of $(E_0),\cdots,(E_{n-1})$ with coefficients which are functions of the coordinates of $U(P)$. For a finite value of $u=r+2$, these coefficients will be functions of class C^{r-n+1}. By differentiation of these relations, it is therefore possible also to express each of the sets $(E_{n+1}),\cdots,(E_{r+1})$ linearly and homogeneously in the $(E_0),\cdots,(E_{n-1})$ with coefficients which are continuous functions of the coordinates of $U(P)$. Hence the matrix M has the constant rank ρ in $U(P)$. Hence P is a regular point of the space S as defined in §2. Similarly, if S is analytic or of class C^∞, it follows that the matrix M has the rank ρ in $U(P)$ and hence that P is a regular point in accordance with the definition of §2. But $\rho \leqq n-m$ in consequence of the assumption that $R_m=0$ over S. Hence the matrix M has a constant rank less than or equal to $n-m$ in each component K. Conversely, if M has a constant rank less than or equal to $n-m$ in each component K, it follows that $R_m=0$ is satisfied over S. If Theorem II is taken into account, the following theorem can now be stated.

THEOREM V. *A necessary and sufficient condition that there exist at least m independent fields of parallel vectors where m has one of the values $1, 2, \cdots, n$ in an arbitrary open, connected, and simply connected point set O contained in an arbitrary component K of an affinely connected space S is that $R_m=0$ over S.*

UNIVERSITY OF CALIFORNIA AT LOS ANGELES

RANDOM SAMPLING IN THE EVALUATION OF A LEBESGUE INTEGRAL

C. R. ADAMS AND A. P. MORSE

The purpose of this note is to show that in the evaluation of a Lebesgue integral a sort of random sampling scheme is permissible. It is possible, therefore, that the results may have application to questions of probability or statistics.

The chief positive result established here is embodied in Theorem 4. With the object of indicating clearly the question at issue, however, we formulate two particular cases in the preliminary Theorems 1 and 2. In connection with the first of these particular cases a negative result, exhibiting the need for a limitation which is placed upon the sampling process, will be shown. Theorem 3, a third particular case of Theorem 4, will be established as an aid to the proof of Theorem 4.

Let $x(t)$ be summable on the interval $0 \leqq t \leqq 1$, that is, $x \in L([0, 1])$; let k be a fixed integer greater than or equal to 1; let $[0, 1]$ be divided into n equal parts ($n = 1, 2, 3, \cdots$) each of length $1/n$; and let the ith one ($i = 1, 2, \cdots, n$) of these parts in turn be divided into k equal parts; let m_{n,i,j_i}, ($i = 1, 2, \cdots, n$; $j_i = 1, 2, \cdots, k$ for each i), designate the L-integral mean of $x(t)$ on the j_ith of the k equal parts into which the ith subinterval of $[0, 1]$ has been divided; and form the "Riemann-Lebesgue" sum

$$kS_n = \sum_{i=1}^{n} m_{n,i,j_i}/n.$$

Clearly this sum is multiple-valued, of multiplicity k^n.

THEOREM 1. *Under the conditions stated, we have*

$$\lim_{n\to\infty} kS_n = \int_0^1 x(t)dt.$$

This can be proved by showing that if S_n' and S_n'' denote arbitrary values of S_n obtainable by some choices of the j_i, we have

$$|S_n' - S_n''| \leqq \int_0^{1-1/(kn)} |x(t + 1/(kn)) - x(t)|\, dt,$$

which tends to zero with $1/n$. The limit of kS_n may then easily be identified with $\int_0^1 x(t)dt$.

Before proceeding to the second preliminary theorem we show that, if k is allowed to be a function of n, $k(n)$, with

$$\limsup_{n\to\infty} k(n) = \infty,$$

there always exists an $x \in L([0,1])$ (in fact, a *bounded* summable function) for which Theorem 1 does not hold. Indeed, under this condition on $k(n)$, there exists a non-decreasing sequence $\{N_n\}$ such that $k(N_n)$ is greater than or equal to 2^{n+1} for every n. Let A_n be the subset of $[0,1]$ defined by*

$$A_n = \sum_{i=0}^{N_n-1} [i/N_n,\ i/N_n + 1/(N_n k(N_n))].$$

Clearly A_n has measure less than or equal to $1/2^{n+1}$ for each n, and $A = \sum_{n=1}^{\infty} A_n$ has measure less than or equal to $1/2$. Let $x(t)$ be the characteristic function of the set A. Then if we take j_i always as 1, it is clear that

$$k(N_n)S_{N_n} = \sum_{i=1}^{N_n} m_{N_n,i,1}/N_n = 1 \qquad \text{for every } n,$$

whence, as $n\to\infty$,

$$k(N_n)S_{N_n} \to 1 > 1/2 \geqq \int_0^1 x(t)dt.$$

Incidentally, it may be noted that there exist other selections of n and the j_i under which $k(n)S_n$ tends to a limit less than or equal to

$$\int_0^1 x(t)dt.$$

On the other hand, we observe that for each fixed $x \in L([0,1])$, there exists a function $k(n)$ with $\lim_{n\to\infty} k(n) = \infty$ for which we do have

$$\lim_{n\to\infty} k(n)S_n = \int_0^1 x(t)dt.$$

For, by Theorem 1, to each integer $m>0$ corresponds an integer $N_m>0$ such that

$$\left| mS_n - \int_0^1 x(t)dt \right| < 1/m$$

* The symbol $[a, b]$ stands for the closed interval $a \leqq t \leqq b$.

for all $n > N_m$. If, in addition, the N_m are chosen to satisfy the condition $N_m < N_{m+1}$ for every m, we may define $k(n) = n$ for $N_n < n \leqq N_{n+1}$, $(n = 1, 2, 3, \cdots)$, and obtain the property asserted.

A complete extension of Theorem 1 to what is usually thought of as a Riemann integral set-up is provided by the following theorem.

THEOREM 2. *Let $x \in L([0, 1])$ and k be any fixed real number greater than or equal to 1; and for $0 < \delta \leqq 1$ let $H(\delta)$ stand for the aggregate of measurable subsets A of $[0, 1]$ defined by the condition $A \in H(\delta)$ if and only if there exist numbers $a_0, a_1, \cdots, a_p$ with $0 = a_0 < a_1 < \cdots < a_p = 1$ such that**

$$|\,[a_{j-1}, a_j) \cdot A\,| = (a_j - a_{j-1})/k, \quad a_j - a_{j-1} < \delta, \quad j = 1, 2, \cdots, p.$$

Then we have

$$\lim_{\delta \to 0} \inf_{A \in H(\delta)} k \int_A x(t)dt = \int_0^1 x(t)dt = \lim_{\delta \to 0} \sup_{A \in H(\delta)} k \int_A x(t)dt.$$

Still further generalization, however, is possible. As a first step toward it we establish the following lemma, in which $C([0, 1])$ stands for the class of functions continuous on $[0, 1]$.

LEMMA. *Let $x \in C([0, 1])$. Corresponding to any $\epsilon > 0$, there exists a $\delta > 0$ such that if $\{A_n\}$ and $\{B_n\}$ are sequences (finite or infinite) of measurable subsets of $[0, 1]$ satisfying the conditions*

$$A_n \subset B_n, \quad 0 < |\,A_n\,|, \quad \text{diameter } (B_n) < \delta, \qquad n = 1, 2, 3, \cdots;$$
$$|\,B_m \cdot B_n\,| = 0, \ m \neq n; \qquad \sum_n B_n = [0, 1],$$

then we have

$$\left| \int_0^1 x(t)dt - \sum_n |\,B_n\,| \cdot \int_{A_n} x(t)dt / |\,A_n\,| \right| < \epsilon.$$

PROOF. There exists $\delta > 0$ such that $|t_1 - t_2| < \delta$ implies $|x(t_1) - x(t_2)| < \epsilon/2$. On each set B_n of diameter less than δ, the integrals†

$$\int_{B_n} x/|\,B_n\,|, \qquad \int_{A_n} x/|\,A_n\,|$$

both lie between the greatest lower and least upper bounds of $x(t)$ on

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* We use $|E|$ to denote the measure of a measurable set E.

† Henceforth we shall often write, for example, $\int_{B_n} x/|B_n|$ in place of $\int_{B_n} x(t)dt/|B_n|$, when such abbreviation seems likely to cause no confusion.

the set B_n, whence these integrals differ by $\epsilon/2$ at most. Therefore we infer

$$\left|\int_0^1 x - \sum_n |B_n| \cdot \int_{A_n} x/|A_n|\right|$$

$$= \left|\sum_n |B_n| \cdot \int_{B_n} x/|B_n| - \sum_n |B_n| \cdot \int_{A_n} x/|A_n|\right|$$

$$\leqq \sum_n |B_n| \cdot \left|\int_{B_n} x/|B_n| - \int_{A_n} x/|A_n|\right| \leqq \frac{\epsilon}{2} \sum_n |B_n| < \epsilon.$$

Remarks. It is now a simple matter to extend this lemma to the case of $x \in R([0, 1])$, where R represents the class of Riemann-integrable functions. For an $x \in R$ can be approximated in the mean within $\epsilon/2$ by a function $y \in C$ with $y(t) \geqq x(t)$ on $[0, 1]$. Then if δ corresponds to $\epsilon/2$ for y in accordance with the lemma, we have

$$\sum_n |B_n| \cdot \int_{A_n} x/|A_n| \leqq \sum_n |B_n| \cdot \int_{A_n} y/|A_n| < \int_0^1 y + \epsilon/2 < \int_0^1 x + \epsilon.$$

Similarly, by approximating x by y from below, we obtain the other half of the desired inequality. Clearly the lemma holds also for any function x which is almost everywhere equal to a function $y \in R([0, 1])$.

That the lemma cannot be extended to the case of $x \in L([0, 1])$ is evident; for whenever x is not essentially bounded, the A_n can be so chosen that the corresponding sum is arbitrarily large, and whenever x is essentially bounded but not almost everywhere equal to a function $y \in R([0, 1])$, the A_n and B_n can be so chosen that the corresponding sum approximates either the essential upper or the essential lower Darboux integral of x.

The following theorem extends Theorem 2 in three distinct directions.

THEOREM 3. *Let $x \in L([0, 1])$ and k be any fixed real number greater than or equal to 1. Corresponding to any $\epsilon > 0$, there exists a $\delta > 0$ such that if $\{A_n\}$ and $\{B_n\}$ are sequences (finite or infinite) of measurable subsets of $[0, 1]$ satisfying the conditions*

$$A_n \subset B_n, \quad 0 < |B_n| \leqq k|A_n|, \quad \text{diameter } (B_n) < \delta, \quad n = 1, 2, 3, \cdots;$$

$$|B_m \cdot B_n| = 0, \quad m \neq n; \qquad \sum_n B_n = [0, 1],$$

then we have

$$\left| \int_0^1 x(t)dt - \sum_n |B_n| \cdot \int_{A_n} x(t)dt/|A_n| \right| < \epsilon.$$

PROOF. Let $y \in C([0, 1])$ be such that $\int_0^1 |x-y| < \epsilon/(2k+2)$, and let δ correspond to $\epsilon/2$ for y in accordance with the lemma. Then we have

$$\begin{aligned}
&\left| \int_0^1 x - \sum_n |B_n| \cdot \int_{A_n} x/|A_n| \right| \\
&\quad = \left| \int_0^1 x - \int_0^1 y + \int_0^1 y - \sum_n |B_n| \cdot \int_{A_n} y/|A_n| \right. \\
&\qquad\qquad \left. + \sum_n |B_n| \cdot \int_{A_n} (y-x)/|A_n| \right| \\
&\quad \leqq \int_0^1 |x-y| + \left| \int_0^1 y - \sum_n |B_n| \cdot \int_{A_n} y/|A_n| \right| \\
&\qquad\qquad + \sum_n |B_n| \cdot \int_{A_n} |x-y|/|A_n| \\
&\quad < \epsilon/(2k+2) + \epsilon/2 + k \int_0^1 |x-y| < \epsilon.
\end{aligned}$$

Finally, we have the following theorem.

THEOREM 4. *Theorem 3 holds for $x \in L(E)$, where E is an arbitrary measurable set, bounded or unbounded.*

PROOF. When E is bounded, the validity of this result may be seen at once. For the above lemma holds if x is *uniformly* continuous on E. And if $E \subset [a, b]$, x may be defined as zero on $[a, b]-E$, and a function $y \in C([a, b])$ approximating x in the mean within $\epsilon/(2k+2)$ on $[a, b]$ is uniformly continuous on E and approximates x in the mean within $\epsilon/(2k+2)$ on E; so that the argument used above in the proof of Theorem 3 applies without change. Moreover, the reasoning shows clearly that the following statement is valid.

(i) *If δ corresponds to x, E, k, ϵ for a bounded set E, and E_1 is any measurable subset of E, the same δ corresponds to x, E_1, k, ϵ.*

To dispose of the case of E unbounded, let H_M stand for the set $[-M, M]$, $M>0$; and $\tilde{H}_M$ for the complement of H_M with respect to the infinite interval $-\infty < t < +\infty$. First, let M be chosen so that

$\int_{E\cdot\bar{H}_M}|x(t)|\,dt<\epsilon/(2k+2)$. Secondly, in accordance with the theorem for a bounded set, let δ, $(\delta<1)$, correspond to x, $E\cdot H_{M+1}$, k, $\epsilon/2$. Thirdly, let $\{A_n\}$ and $\{B_n\}$ be any sequences satisfying the hypotheses; and allow $\sum_n'$ to stand for a sum extended over all values of n for which $|B_n\cdot H_M|>0$, and $\sum_n''$ for the rest of the sum $\sum_n$. Then we have

$$\begin{aligned}
&\left|\int_E x-\sum_n|B_n|\cdot\int_{A_n}x/|A_n|\right| \\
&\quad=\left|\int_{\Sigma_n''B_n}x+\int_{\Sigma_n'B_n}x-\sum_n'|B_n|\cdot\int_{A_n}x/|A_n|\right. \\
&\qquad\left.-\sum_n''|B_n|\cdot\int_{A_n}x/|A_n|\right| \\
&\quad\leqq\int_{\Sigma_n''B_n}|x|+\left|\int_{\Sigma_n'B_n}x-\sum_n'|B_n|\cdot\int_{A_n}x/|A_n|\right| \\
&\qquad+k\int_{\Sigma_n''B_n}|x| \\
&\quad<(k+1)\int_{E\cdot\bar{H}_M}|x|+\epsilon/2<\epsilon
\end{aligned}$$

in consequence of the relation $\sum_n'B_n\subset E\cdot H_{M+1}$, the choice of δ, and the remark (i).

Similar results obtain for a function of several variables, since the above proofs (employing only the uniform continuity of a function continuous over a closed interval and the possibility of approximating in the mean a summable function by a continuous one) are essentially independent of dimensionality.

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A CHARACTERIZATION OF DEDEKIND STRUCTURES*

MORGAN WARD

If Σ is a Dedekind structure,† then for any two elements A and B of Σ, the quotient structures $[A, B]/A$ and $B/(A, B)$ are isomorphic. (Dedekind [2], Ore [3].) I prove here a converse result.

THEOREM. *Let Σ be a structure in which for every pair of elements A and B, the quotient structures $[A, B]/A$ and $B/(A, B)$ are isomorphic. Then if either the ascending or descending chain condition holds in Σ, the structure is Dedekindian.*

This result is comparatively trivial if *both* the ascending and descending chain conditions hold. That some sort of chain condition is necessary may be seen by a simple example. Consider a structure Σ with an all element O_0 and a unit element E_0 built up out of three ordered structures Σ_1, Σ_2, Σ_3 meeting only at O_0 and E_0, so that if $S_u \in \Sigma_u$, then

$$(S_u, S_v) = E_0, \qquad [S_u, S_v] = O_0$$

for $u, v = 1, 2, 3$, $u \neq v$. Then if each Σ_i is a series of the type of the real numbers in the closed interval 0, 1, the quotient structures of any pair $[S_u, S_v]/S_u$, $S_v/(S_u, S_v)$ are obviously isomorphic. But Σ is clearly non-Dedekindian.

The theorem is of some interest in view of the generalizations Ore has given of his decomposition theorems in Ore [4].

It suffices to prove the result under the hypothesis that the descending chain axiom holds in Σ (Ore [3, p. 410]). We formulate this axiom as follows:

(β) *If for any two elements A and B of Σ,*

$$A \supset X_1 \supset X_2 \supset X_3 \supset \cdots \supset B$$

for an infinity of X_i in Σ, all the X_i are equal from a certain point on.

Our proof rests upon several lemmas which we collect here.

LEMMA 1. (Dedekind [2].) *Σ is a Dedekind structure if and only if Σ contains no substructure Σ_0 of order five which is non-Dedekindian.*

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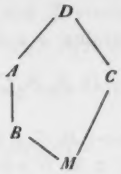
* Presented to the Society, April 15, 1939.

† We use the notation and terminology of Ore's fundamental paper, Ore [3], with the following two exceptions. (i) We write $A \supset B$, $B \subset A$ for Ore's $A \geqq B$, $B \leqq A$. (ii) If A is prime over B (Ore [3, p. 411]), we shall say "A covers B" or "B is covered by A" (Birkhoff [1]) and write $A > B$ or $B < A$.

The type of substructure in question is well known; its diagram is given in the figure. Since we utilize such substructures frequently in our proof, we shall introduce the notation $\{D, A, B, C, M\}$ for Σ_0, writing the all element D and unit element M in the first and last



Σ_0

places in the symbol while the elements A and B where $A \supset B$ occupy the second and third places.

LEMMA 2. (Ore [3].) *If* (β) *holds in the structure* Σ, *then every set of elements of* Σ *which divide a fixed element* A *contains at least one minimal element dividing no other element of the set.*

LEMMA 3. *If* (β) *holds in the structure* Σ, *then for any two distinct elements* A *and* C *of* Σ *such that* C *divides* A, *there exists an element* B *such that* C *divides* B *and* B *covers* A.

For we need only pick a minimal element in the subset of all elements X such that $C \supset X \supset A$, $X \neq A$.

The following lemma is obvious:

LEMMA 4. *Let* Σ *be a structure in which*

$$[A, B]/A \cong B/(A, B) \tag{ϵ}$$

for every A, B *of* Σ. *Then* $[A, B]$ *covers* A *if and only if* B *covers* (A, B).

LEMMA 5. *Let* Σ *be a structure in which* (ϵ) *holds. Then if* A *covers* B *and* M *is any other element of* Σ, *either* $[M, A]$ *equals* $[M, B]$ *or* $[M, A]$ *covers* $[M, B]$.

For clearly $[M, A] \supset [M, B]$. Since $A \supset (A, [M, B]) \supset B$ and $A > B$, either $(A, [M, B]) = A$ or $(A, [M, B]) = B$. If $(A, [M, B]) = A$, then $[M, B] \supset A \supset [M, A]$, so that $[M, B] = [M, A]$. If $(A, [M, B]) = B$, then $A > (A, [M, B])$. Hence by Lemma 4, $[A, [M, B]] > [M, B]$. But since $A \supset B$,

$$[A, [M, B]] = [M, A].$$

Our final lemma is the dual of Lemma 5.

LEMMA 6. *Let Σ be a structure in which* (ϵ) *holds. Then if A covers B and M is any other element of Σ, either (M, A) equals (M, B) or (M, A) covers (M, B).*

We shall prove our theorem indirectly. Assume that conditions (β) and (ϵ) hold in the structure Σ, but that Σ is non-Dedekindian. Then by Lemma 1, Σ contains a non-Dedekindian substructure

$$\Sigma_0 = \{D, A, B, C, M\}$$

of order five.*

We may assume that A covers B. For by Lemma 3, there exists an element N of Σ such that $A \supset N$, $N > B$. Thus

$$[A, C] \supset [N, C] \supset [B, C], \qquad (A, C) \supset (N, C) \supset (B, C);$$

that is, $[N, C] = D$, $(N, C) = M$. Hence $\{D, N, B, C, M\}$ is a non-Dedekindian substructure where $N > B$.

We assume henceforth that A covers B. Since $[A, C] = D$, $(A, C) = M$, and $[B, C] = D$, $(B, C) = M$, $D/C \cong A/M$, and $D/C \cong B/M$ by (ϵ). Hence $A/M \cong B/M$. But B lies in A/M and $A > B$. Since A corresponds to B under the isomorphism, *there exists an element in B/M covered by B.* Denote it by B_1. Then

$$B > B_1 \supset M. \tag{1}$$

Since $B \supset B_1 \supset M$, $(B, C) \supset (B_1, C) \supset (M, C)$ or $(B_1, C) = M$. Consider next the union $D_1 = [B_1, C]$. Since $B > B_1$, by Lemma 5 either $[B, C] = [B_1, C]$ or $[B, C] > [B_1, C]$; that is, either $D = D_1$ or $D > D_1$.

If $D = D_1$, then on writing A_1 for B, we obtain a non-Dedekindian substructure $\{D_1, A_1, B_1, C, M\}$ in which $A_1 > B_1$.

Now assume that $D > D_1$. Clearly $[A, D_1] = [B, D_1] = D$. Consider the crosscut (B, D_1). Since $B > B_1$, by Lemma 6, either $(B, D_1) = (B_1, D_1)$ or $(B, D_1) > (B_1, D_1)$. That is, since $B \supset (B, D_1)$ and $D_1 \supset B_1$, either $(B, D_1) = B_1$ or $(B, D_1) = B$. *We must have* $(B, D_1) = B_1$. For if $(B, D_1) = B$, then $D_1 \supset B$. Since $D_1 \supset C$, we would have $D_1 \supset [B, C]$, $D_1 = D$, contrary to the assumption $D > D_1$.

Consider next the crosscut $A_1 = (A, D_1)$. Since $A > B$, by Lemma 5 either $(A, D_1) = (B, D_1)$ or $(A, D_1) > (B, D_1)$; that is, either $A_1 = B_1$ or $A_1 > B_1$. *We must have* $A_1 > B_1$. For if $A_1 = B_1$, then $\{D, A, B, D_1, B_1\}$ is a non-Dedekindian substructure. But since $[A, D_1] = D$ and $(A, D_1) = B_1$, by (ϵ) $A/B_1 \cong D/D_1$. This isomorphism is impossible, for $A \supset B > B_1$ while $D > D_1$.

Finally, since $A \supset A_1 \supset C$ and $B \supset B_1 \supset C$, $(A_1, C) = (B_1, C) = M$

* The reader will find a structure diagram helpful in following the argument.

while $[A_1, C]=[B_1, C]=D_1$. Thus $\{D_1, A_1, B_1, C, M\}$ is a non-Dedekindian substructure of Σ in which $A_1>B_1$.

We now replace Σ_0 in either case by $\Sigma_1=\{D_1, A_1, B_1, C, M\}$ and obtain a non-Dedekindian substructure $\Sigma_2=\{D_2, A_2, B_2, C, M\}$ where $A_2>B_2$ and

$$B_1>B_2\supset M. \tag{2}$$

On repeating this reasoning, and combining (1), (2), $\cdots$ we obtain a chain

$$B>B_1>B_2>B_3>\cdots\supset M$$

of indefinite length in which all B_i are distinct, contradicting (β).

References

1. G. Birkhoff, *On the combination of sub-algebras*, Proceedings of the Cambridge Philosophical Society, vol. 29 (1933), pp. 441–464.

2. R. Dedekind, *Über die von drei Moduln erzeugte Dualgruppe*, *Werke*, vol. 2, pp. 371–403.

3. O. Ore, *On the foundation of abstract algebra*, I, Annals of Mathematics, (2), vol. 36 (1935), pp. 406–437.

4. ———, *On the theorem of Jordan-Hölder*, Transactions of this Society, vol. 41 (1937), pp. 266–273.

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EXTENSION OF A DISTRIBUTIVE LATTICE TO A BOOLEAN RING*

H. M. MacNEILLE

The problem of imbedding an abstract distributive lattice in a Boolean algebra by an algebraic extension was suggested to the writer by M. H. Stone in 1933. Hausdorff† had already given a solution of this problem for the case where the given distributive lattice was a ring of point sets. A solution for the abstract case was presented by the writer to the Harvard Mathematical Colloquium (1934), included in his doctoral dissertation, and published.‡

In the meantime, Stone§ had discovered that a Boolean algebra is a special type of algebraic ring. This revealed many properties of Boolean algebras to be instances of known ring properties and led the writer to believe that the imbedding of a distributive lattice in a Boolean algebra might be subsumed under some well established algebraic procedure. This was found to be the case.‖ For, if a hypercomplex system is constructed upon the given distributive lattice as a basis with the integers modulo 2 as coefficient field, the resulting ring, reduced by an ideal, is the required extension. This construction is presented in this paper. Aside from the unification it achieves, it is shorter and more elegant than previous solutions.

An algebraic ring¶ in which every element is idempotent with re-

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* Presented to the Society, September 3, 1936, under the title *Extension of a multiplicative system to a Boolean ring.*

† F. Hausdorff, *Mengenlehre*, 2d edition, Leipzig, de Gruyter, 1927, p. 79. Hausdorff's solution uses the assumption that the elements are point sets in defining equality. Garrett Birkhoff has proved (Proceedings of the Cambridge Philosophical Society, vol. 29 (1933), pp. 441–464, Theorem 25.2) that every distributive lattice can be represented by a ring of point sets. Together, the work of Hausdorff and Birkhoff establishes the desired conclusion. The object of Stone's suggestion was to avoid the use of transfinite induction upon which Birkhoff's proof depends.

‡ H. M. MacNeille, *Extensions of Partially Ordered Sets*, doctoral dissertation, Harvard University, 1935; Proceedings of the National Academy of Sciences, vol. 22 (1936), pp. 45–50; Transactions of this Society, vol. 42 (1937), pp. 416–460.

§ M. H. Stone, Proceedings of the National Academy of Sciences, vol. 21 (1935), pp. 103–105.

‖ H. M. MacNeille, this Bulletin, abstract 42-7-316.

¶ For the definition and elementary properties of algebraic rings and hypercomplex systems, see B. L. van der Waerden, *Moderne Algebra*, Berlin, Springer, 1931, vol. 1, chap. 3, and vol. 2, chap. 16. For rings and fields of point sets, see Hausdorff, loc. cit., §17. For Boolean rings, see M. H. Stone, Transactions of this Society, vol. 40 (1936), pp. 37–111, chap. 1. For distributive lattices, see M. H. Stone, Czecho-

spect to multiplication ($aa=a$) is called a Boolean ring. Such a ring is necessarily commutative ($ab=ba$), and addition is modulo 2 ($a+a=0$). A Boolean ring may, but need not, contain a unit for multiplication. In any Boolean ring, a third operation $a \vee b$ is defined by the equation

$$a \vee b = a + b + ab. \tag{1}$$

When the Boolean ring is a field of point sets, $a \vee b$ is the union of a and b. Furthermore, in any Boolean ring, $a \vee b$ enjoys the algebraic properties of point set union and will be referred to as the union of a and b even when a and b are not point sets.

Regarding union and multiplication as the basic operations, every Boolean ring is a distributive lattice. It is natural to ask whether the definition (1) of union can be inverted so as to define addition in terms of union and multiplication. However, one finds this not to be the case. Not every distributive lattice is a Boolean ring.* Closure with respect to addition and multiplication implies the existence of relative complements, while closure with respect to union and multiplication does not. That is, if a and b are elements of a Boolean ring, an element x (namely $b+ab$) exists such that $a \vee x = a \vee b$ and $ax=0$, while the distributive lattice operations of union and multiplication do not require such an element. Thus the problem of imbedding a distributive lattice in a Boolean ring is not vacuous.

Let a distributive lattice K with elements $a, b, c, \cdots$, and operations of union and multiplication be given. Consider the hypercomplex system H generated by K as basis and with the integers modulo 2 as coefficient field. Equal elements in K are regarded as identical. Multiplication of basis elements is taken as the given lattice multiplication. In many cases, the rank of H will be infinite, but elements of H are allowed only a finite number of coefficients different from zero. Elements of H can be represented as sums $(\sum_{i=1}^{n} a_i = a_1+a_2+a_3+\cdots+a_n)$ of a finite number of elements of K, where inclusion in the summation indicates that the coefficient is 1 and omission from the summation indicates that the coefficient is 0. Furthermore, since equal elements in K are regarded as identical, no two summands are equal. Two elements of H are equal if they have

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slovakian Journal for the Advancement of Mathematics and Physics, vol. 67 (1937), no. 1, pp. 1–25, §1, and Birkhoff, *C lattices*, loc. cit. The present paper follows the notation of Stone.

* For example, if union is defined as least upper bound and multiplication as greatest lower bound, the real number system is a distributive lattice but not a Boolean ring.

the same summands. Each summand is called a component of the element. H contains the element all of whose coefficients are 0. This is the zero element in H and will be denoted by 0. The system H is a ring, and every element of H is idempotent with respect to multiplication; therefore we have the following theorem:

THEOREM. *H is a Boolean ring.*

The lattice K is imbedded in H, as each element of K is a basis element of H. Equality and multiplication in K correspond to equality and multiplication in H. If K contains a zero element, it is the element of H with itself as sole component. The zero of H has no components. These two elements form an ideal in H. Reducing H by this ideal identifies the zero of K with the zero of H and has no other effect. From this point on, we assume that, if K has a zero, it has been identified with the zero of H and never occurs as a component of an element of H.

Equation (1) defines union in H, but this definition does not correspond with union as defined in K. This is evident from the fact that the union in H of two elements in K has three components while the union in K of these elements has but one component in H. Let M be the subring of H generated by all elements of the form $(a \vee b)+a+b+ab$, where $a \vee b$ is the union in K of a and b. The subring M is an ideal in H. Let L be the quotient ring H/M. The ring L is a Boolean ring in which K is imbedded and (1) holds when the indicated union is performed in K. That is, the operation of union as given in K corresponds with union as defined in terms of addition and multiplication in L. The reduction of H by M does not introduce new equalities in K. For, if $a \neq b$ in K, then $a+b \neq 0$ in H, and $a+b$ does not belong to M. Therefore, $a+b \neq 0$ in L and $a \neq b$ in L.

It remains to show that L is the smallest Boolean ring in which K can be imbedded. That is, we must show that any reduction of L imposes new equalities on K. As a first step, a standard form is found for the elements of L. An element $\sum_{i=1}^{n} a_i$ of L is said to be in standard form if $a_i a_j = a_j$ whenever $i \leqq j$.

LEMMA. *Every element of L can be expressed in standard form.*

The proof is by induction on n and successive applications of the identity $a+b=(a \vee b)+ab$. Since $(a \vee b)ab=ab$, this identity itself establishes the lemma for $n=2$. Assuming the lemma established for $n=k$, consider $\sum_{i=1}^{k+1} a_i$ where the last k summands are assumed in standard form. Applying the identity k times to pairs of adjacent summands starting with the first and second, and increasing indices

by one each time, one obtains

$$\sum_{i=1}^{k+1} a_i = \sum_{i=1}^{k} (a_1 a_i \vee a_{i+1}) + a_1 a_{k+1}$$

where the right-hand member of the equation is in standard form.

LEMMA. *If* $\sum_{i=1}^{n} a_i$, $(n \neq 0)$, *is in standard form, the assumption that* $\sum_{i=1}^{n} a_i = 0$ *imposes new equalities on* K.

If n is odd, multiplying through by a_n, one obtains $a_n \sum_{i=1}^{n} a_i = a_n = 0$ which implies $a_n = 0$ in K, contrary to hypothesis; and if n is even, multiplying through by a_{n-1}, one obtains $a_{n-1} \sum_{i=1}^{n} a_i = a_{n-1} + a_n = 0$ which implies $a_{n-1} = a_n$ in K, contrary to hypothesis.

The following theorems are immediate consequences of this lemma.

THEOREM. *An element of* L *in standard form is equal to zero only if it is identically zero.*

THEOREM. L *is the smallest Boolean ring in which* K *can be imbedded.*

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SPACES OF UNCOUNTABLY MANY DIMENSIONS*

C. W. VICKERY

Riemann in his *Habilitations Schrift* of 1854 suggested the notion of n-dimensional space (where n is a natural number) as an extension of the notion of three-dimensional euclidean space. Hilbert extended the notion still further by defining a space of a countably infinite number of dimensions. Fréchet† in 1908 defined two other spaces of countably many dimensions, which he called D_ω and E_ω. Tychonoff‡ in 1930 defined a series of spaces of an unlimited number of dimensions and established several of their properties. The present paper undertakes, by generalizing the notions of spaces D_ω and E_ω, to define spaces D^α and E^α, respectively, for each cardinal number $\aleph_\alpha$ representing the number of dimensions. It is shown that every metric space is homeomorphic with a subset of some space D^α. Certain properties of Tychonoff's spaces, here called spaces T^α, are also presented.

1. **Spaces D^α.** For each initial ordinal number ω_α the set of all points of space D^α is the set of all type ω_α sequences $[x_i]_i^{\omega_\alpha}$ of real numbers x_i, such that $0<x_i<1$. Suppose that $[P_i]_i^{\omega_0}$ is a type ω_0 sequence of points of space D^α such that for each i, $P_i=[x_{i,j}]_j^{\omega_\alpha}$; and suppose that $P=[y_j]_j^{\omega_\alpha}$ is a point of space D^α. The sequence $[P_i]_i^{\omega_0}$ is said to *converge* to the point P if and only if it is true that if ϵ is a positive number, there exists a positive integer N_ϵ such that if $i>N_\epsilon$, then $|y_j-x_{i,j}|<\epsilon$ for every value of $j<\omega_\alpha$. The point P is said to be the *sequential limit point* of $[P_i]_i^{\omega_0}$. A point P is said to be a *limit point* of a point set M, provided there exists a sequence of distinct points of M which converges to P. Thus space D^0 is equivalent to space D_ω of Fréchet. For each two distinct points $A=[x_i]_i^{\omega_\alpha}$ and $B=[z_i]_i^{\omega_\alpha}$ of D^α such that for each i, $x_i\neq z_i$, let $D^\alpha_{(A,B)}$, also referred to as *segment* AB, denote the set of all points $P=[y_i]_i^{\omega_\alpha}$ such that $x_i<y_i<z_i$ or $z_i<y_i<x_i$. If there exist constants h and k, $(h<k)$, such that for each i, $x_i=h$ and $y_i=k$, the notation $D^\alpha_{(h,k)}$ is used. It is evident that $D^\alpha_{(h,k)}$ is homeomorphic with D^α for every value of α, h, and k; but there exist points A and B such that $D^\alpha_{(A,B)}$ is not homeomorphic with D^α.

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* Presented to the Society, September 12, 1935, and October 26, 1935 (under the title *Concerning spaces of uncountably many dimensions*).

† M. Fréchet, *Essai de géométrie analytique à une infinité de coordinées*, Nouvelles Annales de Mathématique, (4), vol. 8 (1908), pp. 97–116. Also *Les Espaces Abstraits*, Paris, 1928, pp. 81–84, 97–99.

‡ Mathematische Annalen, vol. 102 (1930), pp. 544–561.

THEOREM 1.1. *Space D^α is metric, for every value of α.*

Suppose that $A=[x_i]_i^{\omega_\alpha}$ and $B=[y_i]_i^{\omega_\alpha}$ are points of D^α. Let the distance from A to B, $d(A, B)$, be the least upper bound of the numbers of the sequence $[|y_i-x_i|]_i^{\omega_\alpha}$. This definition can be shown to satisfy the conditions of Fréchet for metric spaces, as in the case of space D_ω.

THEOREM 1.2. *In space D^α there exists a point set of power $2^{\aleph_\alpha}$ which has no limit point.*

Let M denote the set of all points P such that each of the coordinates of P is either $1/4$ or $1/2$. Then M is of power $2^{\aleph_\alpha}$. If A and B are distinct points of M, then $d(A, B)=1/4$. Hence no point of M is a limit point of M. But M is closed since any limit point of M has each of its coordinates either $1/4$ or $1/2$. Hence M has no limit point.

THEOREM 1.3. *Space D^α is not $\aleph_\varkappa$-separable* if $\aleph_\varkappa<2^{\aleph_\alpha}$.*

Suppose the theorem is false. There exists a cardinal number $\aleph_\beta<2^{\aleph_\alpha}$ such that there exists a subset K of power $\aleph_\beta$, everywhere dense in D^α. Hence D^α is $\aleph_\beta$-completely separable.* Let G denote a collection of power $\aleph_\beta$ of domains with respect to which D^α is $\aleph_\beta$-completely separable. By Theorem 1.2 there exists a point set of power $2^{\aleph_\alpha}$ which has no limit point. For every point P of M there exists a domain of G which contains P and no other point of M; and this involves a contradiction.

THEOREM 1.4. *If M is an $\aleph_\alpha$-separable metric space, then M is homeomorphic with a subset of space D^α.*

This theorem is a generalization of a theorem of Urysohn,† and the proof is a generalization of that of Urysohn.

There exists a distance function $d(A, B)$ such that if A and B are distinct points of M, then $0<d(A, B)<1$. Let K denote a subset of M, of power $\aleph_\alpha$, everywhere dense in M. Let $[K_i]_i^{\omega_\alpha}$ denote a type ω_α sequence of all the elements of K. For each point P of M let

$$\phi(P) = [1/2 + 1/2d(P, K_i)]_i^{\omega_\alpha}.$$

The correspondence is one-to-one. To show this, it is sufficient to

* For a definition of this term see C. W. Vickery, *Spaces in which there exist uncountable convergent sequences of points*, Tôhoku Mathematical Journal, vol. 40 (1934), p. 11. Cf. Keitaro Haratomi, *Über höherstufige Separabilität und Kompaktheit*, Japanese Journal of Mathematics, vol. 8 (1931), pp. 113–141; vol. 9 (1932), pp. 1–18.

† P. Urysohn, *Les classes (D) separables et l'espace Hilbertien*, Comptes Rendus de l'Académie des Sciences, vol. 178 (1924), p. 65.

show that if A and B are distinct points, $\phi(A) \neq \phi(B)$. Suppose that A and B are distinct points. There exists a point K_j of K such that

$$d(A, K_j) < 1/3d(A, B).$$

But

$$d(A, K_j) + d(K_j, B) \geqq d(A, B).$$

Hence $d(A, K_j) < d(K_j, B)$, and hence $\phi(A) \neq \phi(B)$.

The correspondence is also bicontinuous. Suppose that ϵ is a positive number and P and Q are points of M such that $d(P, Q) < \epsilon$. Let $\phi(P) = [x_i]_i^{\mu_\alpha}$ and $\phi(Q) = [y_i]_i^{\mu_\alpha}$. Suppose further that K_i is an element of $[K_i]_i^{\mu_\alpha}$. Then

$$d(P, K_i) + d(P, Q) \geqq d(Q, K_i), \quad d(Q, K_i) + d(P, Q) \geqq d(P, K_i).$$

Since $d(P, Q) < \epsilon$, it follows that

$$|\, d(Q, K_i) - d(P, K_i) \,| < \epsilon$$

and hence

$$|\, x_i - y_i \,| < \epsilon.$$

Since this relation holds for every value of i, it follows that if P is a limit point of a point set H, then $\phi(P)$ is a limit point of $\phi(H)$.

Suppose now that δ is a positive number and P and Q are points of M such that $d(P, Q) > \delta$. Let $\phi(P) = [x_i]_i^{\mu_\alpha}$, $\phi(Q) = [y_i]_i^{\mu_\alpha}$, and $d(P, Q) = \delta + \eta$. There exists an ordinal number x such that $d(Q, K_x) < \eta/2$. It follows that

$$|\, d(P, K_x) - d(P, Q) \,| < \eta/2.$$

Hence

$$d(P, K_x) > \delta + \eta/2,$$

hence

$$d(P, K_x) - d(Q, K_x) > \delta,$$

and hence

$$|\, x_x - y_x \,| > \delta/2.$$

Therefore if P is not a limit point of a point set H, then $\phi(P)$ is not a limit point of $\phi(H)$.

Note. Since space D^α is not $\aleph_x$-separable for any cardinal number $\aleph_x < 2^{\aleph_\alpha}$, the power of the space, it thus appears that there is a relation between the dimensionality of a metric space and its type of

separability. The type of separability does not determine the dimensionality, but it fixes an upper bound to it.

THEOREM 1.5. *In order that a space S be metric, it is necessary and sufficient that there exist a cardinal number $\aleph_\alpha$ such that S is homeomorphic with a subset of space D^α.*

This follows with the help of Theorems 1.3, 1.4, and the fact that every metric space is $\aleph_\varepsilon$-separable for some cardinal number $\aleph_\varepsilon$.

DEFINITION. *A collection G of real-valued functions defined over a point set M is said to be equi-continuous provided it is true that if P is a point of M and ϵ is a positive number, there exists an open subset D of M containing P such that if f is a function of G, the oscillation of f on D is less than ϵ.*

THEOREM 1.6. *In order that a Hausdorff topological space S be homeomorphic with a subset of space D^α, it is necessary and sufficient that there exist an equi-continuous collection G, of power $\aleph_\alpha$, of functions defined over S, whose values are real numbers between 0 and 1, such that if P is a point and H is a closed point set not containing P, there exists a positive number δ such that if Q is a point of H, there exists a function f of G such that*

$$|f(Q)-f(P)|>\delta.$$

The condition is sufficient. Let $\gamma=[f_i]_i^{\omega_\alpha}$ denote a type ω_α sequence of the elements of G. For each point P, let $\phi(P)=[f_i(P)]_i^{\omega_\alpha}$. Then ϕ represents a one-to-one continuous transformation of S into a subset of space D^α.

The condition is also necessary. Suppose that $\phi(P)$ is a one-to-one continuous transformation of S into a subset of space D^α. For each ordinal number $\nu<\omega_\alpha$ and each point P, where $\phi(P)=[x_i]_i^{\omega_\alpha}$, let $f_\nu(P)=x_\nu$. Let G denote the collection of all functions f_ν so defined. Then G is the required collection.

2. **Spaces E^α.** If the conditions, employed in defining spaces D^α, for the convergence of a sequence of points are made less restrictive so that it is no longer required that the sequences of coordinates converge *uniformly*, spaces E^α are defined. Using the notation of §1, the sequence of points $[P_i]_i^{\omega_0}$ is said to *converge* to the point P if and only if it is true that if j is an ordinal number less than ω_α and ϵ is a positive number, then there exists a positive integer $N_{\epsilon,j}$ such that if $i>N_{\epsilon,j}$ then $|y_j-x_{i,j}|<\epsilon$. Space E^0 is thus equivalent to space E_ω of Fréchet. Every segment $E^\alpha_{(A,B)}$ is homeomorphic with E^α.

THEOREM 2.1. *There exists in space E^1 a compact, non-separable point set.*

Let H denote the set of all points $P=[x_j]_j^{\omega_1}$ such that for each j, $1/4\leqq x_j\leqq 1/2$ and such that there exists an ordinal number $\delta_P<\omega_1$ such that if $j>\delta_P$, then $x_j=1/4$. This set is compact. Suppose that $[P_i]_i^{\omega_0}$ is a type ω_0 sequence of points $P_i=[x_{i,j}]_j^{\omega_1}$ of H; this sequence has a convergent subsequence. For each point P_i there exists an ordinal number δ_i such that if $j>\delta_i$ then $x_{i,j}=1/4$. The sequence of elements δ_i is a type ω_0 sequence; let μ denote the smallest ordinal number greater than all the elements of this sequence. Then $\mu<\omega_1$. Let f denote a one-to-one correspondence between the ordinal numbers less than ω_0 and the ordinal numbers less than μ. Let K denote the set of all points $Q=[y_i]_i^{\omega_1}$ such that if $i>\mu$ then $y_i=1/4$. For each point $Q=[y_i]_i^{\omega_1}$ of K let $\phi(Q)=[y_j]_{f(j)}^{\omega_0}$. Then ϕ represents a one-to-one bicontinuous transformation of the points of K into a subset of space E^0 such that if z_h, $(h<\omega_0)$, is a coordinate of a point of $\phi(H)$, then $1/4\leqq z_h\leqq 1/2$. But in space E^0 the set M of all points whose coordinates are greater than or equal to $1/4$ and less than or equal to $1/2$ is closed and compact. There exists a subsequence of $[\phi(P_i)]_i^{\omega_0}$, namely $[\phi(P_{i_n})]_{i_n}^{\omega_0}$, which converges to a point T of M. Hence $[P_{i_n}]_{i_n}^{\omega_0}$ converges to $\phi^{-1}(T)$.

The point set H is not separable. Suppose that it is separable. Then there exists a countable everywhere dense subset M. There exists an ordinal number $\xi<\omega_1$ such that if $P=[x_i]_i^{\omega_1}$ is a point of M, then if $i>\xi$, $x_i=1/4$. But no point which fails to satisfy this condition is a limit point of M, and since there are uncountably many such points, this leads to a contradiction.

THEOREM 2.2. *Space E^α is not metric for $\alpha\geqq 1$.*

By Theorem 2.1, E^1 contains a compact, non-separable subset H. Hence if $\alpha\geqq 1$, E^α contains a compact, non-separable subset. But every compact subset of a metric space is separable. Hence E^α is not metric for $\alpha\geqq 1$.

THEOREM 2.3. *If S is a normal, $\aleph_\alpha$-completely separable, topological space (of Hausdorff), then S is homeomorphic with a subset of the space E^α.*

This theorem is a generalization of a theorem of Urysohn* to the effect that every normal, completely separable, Hausdorff topological

* See W. Sierpiński, *Introduction to General Topology*, translated by C. C. Krieger, Toronto, 1934, p. 92.

space is homeomorphic with a subset of Hilbert space. Tychonoff has established a similar theorem for spaces T^α.

THEOREM 2.4. *If S is a normal topological space (of Hausdorff), there exists a cardinal number $\aleph_\alpha$ such that S is homeomorphic with a subset of space E^α.*

This follows with the help of Theorem 2.3 and the fact that S is $\aleph_\alpha$-completely separable for some value of $\aleph_\alpha$.

3. **Spaces T^α.** By a further extension of the notion of limit point, Tychonoff's spaces, here called spaces T^α, are produced. In order to define limit point in these spaces, the notion of region is introduced. Suppose that G_α denotes the collection of all finite sets of ordinal numbers $\nu<\omega_\alpha$. Then G_α is of power $\aleph_\alpha$. Let $H_\alpha=[h_i]_i^{\omega_\alpha}$ denote a type ω_α sequence of all the elements of G_α. If ν is an ordinal number less than ω_α, r a positive real number, and $A=[a_i]_i^{\omega_\alpha}$ a point, let $R_{(A,\nu,r)}$ denote the set of all points $P=[x_i]_i^{\omega_\alpha}$ such that if i is an element of h_ν, then $(a_i-r)<x_i<(a_i+r)$. Then $R_{(A,\nu,r)}$ is called a *region*. The region $R_{(A,\nu,r)}$ is said to have *center* at A, *constrictions* at h_ν, and *radius* r. For all values of A, ν, and r, $R_{(A,\nu,r)}$ is an open set in space E^α. If we now adopt the definition that a point P is a *limit point* of a point set M if and only if every region that contains P contains a point of M distinct from P, we find that there exist point sets M and points P such that P is a limit point of M in this sense, but not in the sense of spaces E^α. This new definition thus constitutes an extension of the notion of limit point and the resulting spaces are designated by T^α. For example, consider the point set M in space T^1 consisting of all points $Q=[x_{i,j}]_j^{\omega_1}$, where $x_{i,j}=1/4$ for $j<i$ and $x_{i,j}=1/2$ for $j\geqq i$. The point set M does not have the point $P=[y_j]_j^{\omega_1}$ (where for each j, $y_j=1/4$) as a limit point in space E^1. However, P is a limit point of M in space T^1.

THEOREM 3.1. *For each cardinal number $\aleph_\alpha$, space T^α is a distributive space $\Gamma_{\aleph_\alpha}$.*

A space is said to be a space $\Gamma_{\aleph_\alpha}$ provided it satisfies Axiom $1_{(\aleph_\alpha)}$, and a distributive space provided it satisfies Axiom 2 of the author's paper *Spaces in which there exist uncountable convergent sequences of points.**

It will be shown that there exists a family F of power $\aleph_\alpha$ satisfying the conditions of Axiom $1_{(\aleph_\alpha)}$. For each ordinal number $\nu<\omega_\alpha$ and each natural number n, let $G_{(\nu,n)}$ denote the collection of all regions of

* C. W. Vickery, op. cit., pp. 12–13.

radius $1/n$ having restrictions at h_r. Let F denote the family of all collections $G_{(r,n)}$. It can easily be shown that F is the required family. Axiom 2 is evidently satisfied.

Note. If in space T^a the point P be said to be the *sequential limit point* of a type ω_a sequence of points $[P_i]_i^{\omega_a}$ if and only if it is true that if R is a region containing P, then R contains a residue of sequence $[P_i]_i^{\omega_a}$, there then exist type ω_a convergent sequences of points. Thus we have been led by a series of apparently natural definitions to the existence of uncountable convergent sequences of points in certain spaces of uncountably many dimensions.

AUSTIN, TEX.

NOTE ON THE LOCATION OF ZEROS OF THE DERIVATIVE OF A RATIONAL FUNCTION WHOSE ZEROS AND POLES ARE SYMMETRIC IN A CIRCLE*

J. L. WALSH

1. **Introduction.** The most general function which effects a 1-to-m conformal transformation of the interior of the unit circle $|z| = 1$ onto itself is of the form

$$r(z) = \lambda \prod_{k=1}^{m} \frac{z - \alpha_k}{1 - \bar{\alpha}_k z}, \quad |\alpha_k| < 1, \quad |\lambda| = 1; \tag{1}$$

so the location of the zeros of the derivative $r'(z)$ is of considerable interest. The zeros and poles of $r(z)$ are symmetric in the unit circle. Moreover a typical transcendental function bounded in the unit circle is the Blaschke product (assumed convergent)

$$B(z) = \prod_{k=1}^{\infty} \frac{\bar{\alpha}_k}{|\alpha_k|} \frac{z - \alpha_k}{\bar{\alpha}_k z - 1}, \tag{2}$$

which is the limit for $|z| < 1$ of a sequence of functions each of form (1). It is of some significance in studying the behavior of $B(z)$ to know exactly or approximately the zeros of $B'(z)$. The object of the present note is to give some fairly simple but elegant results on the derivatives of both $r(z)$ and $B(z)$. Application is made also to the critical points of certain harmonic functions.

2. **Derivative of a rational function.** We first obtain the following result:

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* Presented to the Society, December 30, 1938.

THEOREM 1. *Let a circle K lie interior to C: $|z|=1$ and contain in its interior all the points $\alpha_1, \alpha_2, \cdots, \alpha_m$. Then K contains in its interior precisely $m-1$ zeros of $r'(z)$, where $r(z)$ is defined by* (1). *The inverse with respect to C of the interior of K contains all the remaining finite zeros of $r'(z)$.*

The zeros α_k of $r(z)$ and the poles $1/\bar{\alpha}_k$ of $r(z)$ are mutually inverse in C, so the poles of $r(z)$ lie exterior to K, and the conclusion follows from the results due to Bôcher and the present writer.*

If the points $\alpha_1, \alpha_2, \cdots, \alpha_m$ are given, Theorem 1 enables us, by the use of a number of different circles K, to obtain a region interior to C containing in its interior both all the α_k and all the $m-1$ zeros of $r(z)$ interior to C. Still another result is somewhat more specific:

THEOREM 2. *If a circular region G bounded by a circle Γ orthogonal to C: $|z|=1$ contains in its interior no point α_k, then the region G contains in its interior no finite zero of $r'(z)$, where $r(z)$ is defined by* (1).

In Theorem 2 (and in similar situations below) such a circle as Γ may be a straight line. In proving the theorem it will be a convenience to have for reference the following lemma, whose proof is immediate:

LEMMA 1. *In the field of force due to n unit particles $Q_1, Q_2, \cdots, Q_n$ each repelling with a force equal to the inverse distance, the force at a point P may be found as follows. Let Q_k' be the inverse of the point Q_k in the unit circle whose center is P. Let Q' be the center of gravity of the points $Q_1', Q_2', \cdots, Q_n'$. Then the force at P due to the n particles Q_k is n times the vector $\overline{Q'P}$.*

The finite zeros of $r'(z)$ not multiple zeros of $r(z)$ are the positions of equilibrium in the field of force due to equal repelling particles each of mass $+1$ situated at each of the points α_k, and equal repelling particles each of mass -1 (that is, attracting particles of mass $+1$) situ-

* Bôcher (Proceedings of the American Academy of Arts and Sciences, vol. 40 (1906), pp. 469–484) proves: *If two circular regions T_1 and T_2 have no point in common and contain respectively the roots of two binary forms f_1 of degree p_1 and f_2 of degree p_2, then T_1 and T_2 contain respectively p_1-1 and p_2-1 roots of the jacobian of f_1 and f_2.* A circular region is here and below a closed region of the extended plane bounded by a circle or line.

Bôcher's result was applied by Walsh (Transactions of this Society, vol. 19 (1918), pp. 291–298) to the study of the derivatives of a rational function: *If two circular regions T_1 and T_2 have no point in common and contain respectively all the zeros and all the poles of a rational function $f(z)$ of degree n, then all of the zeros of the derivative $f'(z)$ lie in T_1 and T_2 except that there are two additional zeros of $f'(z)$ at infinity if $f(z)$ has no pole there. Except for these two possible zeros, there are precisely $n-1$ zeros of $f'(z)$ in T_1, and if $f(z)$ has no multiple poles, there are precisely $n-1$ zeros of $f'(z)$ in T_2.*

ated at each of the points $1/\bar{\alpha}_k$; the law of force is that of the inverse distance.* Under the hypothesis of Theorem 2, let the point P lie in the region G. The inverses of C and Γ in the unit circle whose center is P are circles C' and Γ', which are orthogonal to each other; Γ' is necessarily a proper circle; the circle C' separates the images Q_k' of the α_k from the images Q_k'' of the $1/\bar{\alpha}_k$; the points Q_k' and Q_k'' lie in the image of the complement of G, namely the *interior* of Γ'; the point P may or may not lie interior to Γ'. For convenience in exposition, let us orient the plane so that both C' and Γ' are symmetric in a vertical line (that is to say, so that the two points of intersection of C' and Γ' lie on a horizontal line), with each point Q_k' *above* the corresponding point Q_k''; of course Q_k' and Q_k'' are mutually inverse in C'. Then the center of gravity Q' of the points Q_k' lies *above* the center of gravity of the points Q_k''. Consequently, (Lemma 1) the resultant of the attractive forces is not equal and opposite to the resultant of the repelling forces; the point P cannot be a position of equilibrium in the field of force; the point P cannot be a multiple zero of $r(z)$, hence cannot be a zero of $r'(z)$; so Theorem 2 is established.

An immediate consequence of Theorem 2 is the following corollary:

COROLLARY. *Under the hypothesis of Theorem* 2, *if all the points* α_k *lie on* Γ, *then all the finite roots of* $r'(z)$ *also lie on* Γ.

In any given situation, there are a number of circles Γ of the kind required in Theorem 2 that can be drawn to delimit a region in which the zeros of $r'(z)$ lie, so we have the theorem:

THEOREM 3. *Let the points* α_k *be given interior to* C: $|z|=1$, *and let* Π *be the smallest (closed) curvilinear polygon interior to* C *bounded by arcs of circles* Γ *orthogonal to* C, *such that* Π *contains each of the points* α_k, *and where each* Γ *bounds a circular region containing in its interior no* α_k. *Then* Π *contains on or within it all the zeros of* $r'(z)$ *interior to* C, *where* $r(z)$ *is given by* (1).

Theorem 3 is the precise non-euclidean analogue of the classical theorem of Lucas that all the zeros of the derivative $p'(z)$ of a polynomial $p(z)$ lie in the smallest convex polygon containing all the roots of $p(z)$. For the function $w=r(z)$ maps $|z|<1$ conformally in a 1-to-m manner onto $|w|<1$, and is the most general function defining such a map, just as $w=p(z)$ maps $|z|<\infty$ onto $|w|<\infty$ conformally in a 1-to-m manner, and is the most general function defining such a map. The derivatives of both $r(z)$ and $p(z)$ vanish precisely $m-1$

* See Bôcher, loc. cit., Walsh, loc. cit.

times in the regions involved. The regions specified in Theorem 3 and Lucas's theorem are both the smallest convex sets containing the roots of the given function, in the respective senses of non-euclidean and euclidean geometry. The two theorems are respectively invariant under arbitrary one-to-one conformal transformation onto itself of the non-euclidean plane $|z| < 1$ and of the euclidean plane $|z| < \infty$.

An arbitrary simply connected region R of the w-plane with more than one boundary point can be mapped conformally onto $|z| < 1$; a function $f(w)$ analytic in R, whose modulus is continuous in the corresponding closed region and constant on the boundary but not in R, corresponds under the map to a constant multiple of a function $r(z)$ as defined by (1). Thus Theorem 3 yields the result that if non-euclidean geometry is defined in R by means of the conformal map onto $|z| < 1$, then *the zeros of $f'(w)$ in R lie in the smallest non-euclidean convex polygon containing the zeros of $f(w)$ in R*; it is a theorem due to Denjoy that the number of zeros of $f'(w)$ in R is one less than the number of zeros of $f(w)$ in R. Theorems 5–10 may similarly be generalized at once by a conformal map.

If fixed points α_k interior to $|z| = r$ are given, the most general function analytic in $|z| < r$, vanishing precisely in the points α_k, whose modulus is continuous in $|z| \leqq r$ and equal to r^n on $|z| = r$, is given by

$$f(z) \equiv \lambda \prod_{k=1}^{n} \frac{z - \alpha_k}{1 - \bar{\alpha}_k z / r^2}, \qquad |\lambda| = 1;$$

equation (1) is the special case $r = 1$. When r becomes infinite, $f(z)$ approaches the function $p(z) = \lambda \prod_{k=1}^{n} (z - \alpha_k)$; the derivative $f'(z)$ approaches $p'(z)$; the zeros of $f'(z)$ in $|z| < r$ approach the zeros of $p'(z)$; a variable circle through two fixed points z independent of r and orthogonal to $|z| = r$ has a radius which becomes infinite with r, for the center of such a variable circle lies exterior to $|z| = r$. Thus Lucas's theorem is a limiting case of Theorem 3.

3. **Both zeros and poles interior to** C. Lucas's theorem admits of an extension to rational functions:*

* Compare the theorems of Bôcher and Walsh already quoted; also M. B. Porter, Proceedings of the National Academy of Sciences, vol. 2 (1916), pp. 247–248; Walsh, Transactions of this Society, vol. 24 (1922), pp. 31–69; Marden, Transactions of this Society, vol. 32 (1930), pp. 658–668. The theorems of Bôcher and Walsh previously quoted are concerned with *all* the zeros and poles of the functions of the hypothesis, whereas Theorem 4 involves explicitly only the *finite* zeros and poles. But the former results contain the essential content of Theorem 4.

THEOREM 4. *Let $r(z)$ be a rational function of z with the finite zeros $\alpha_1, \alpha_2, \cdots, \alpha_m$ and the finite poles $\beta_1, \beta_2, \cdots, \beta_n$. If L is a line separating each α_j from each β_k, then no finite zero of $r'(z)$ lies on L. Consequently, if such lines L exist, all the finite zeros of $r'(z)$ lie in two unbounded convex point sets which are separated by every L and which contain respectively the points α_j and the points β_k.*

If all the points α_j and all the points β_k lie on a line, and if no α_j separates a pair of the β_k and no β_k separates a pair of the α_j, then all finite zeros of $r'(z)$ lie on that line, and lie on the smallest infinite segments containing respectively all the α_j and none of the β_k, and all of the β_k and none of the α_j.

Theorem 4 has a precise analogue in the non-euclidean case:

THEOREM 5. *Let $r(z)$ be a rational function of z with the zeros $\alpha_1, \alpha_2, \cdots, \alpha_m$ and poles $\beta_1, \beta_2, \cdots, \beta_n$ interior to C: $|z| = 1$, and the poles $1/\bar{\alpha}_k$ and zeros $1/\bar{\beta}_k$ exterior to C, and having no other zeros or poles. If L is a circle orthogonal to C, and if L separates each α_j from each β_k, then no zero of $r'(z)$ lies on L interior to C. Consequently, if a circle L exists, all the zeros of $r'(z)$ interior to C lie in two closed point sets Π_1 and Π_2 which are bounded by circles orthogonal to C each of which separates all the α_j not lying on it from all the β_k not lying on it; these two point sets Π_1 and Π_2 are separated by every L; they contain, respectively, all the α_j and all the β_k.*

If all the points α_j and all the points β_k lie on a circle K orthogonal to C, and if no α_j lies on an arc of K interior to C bounded by a pair of the β_k, and if no β_k lies on an arc of K interior to C bounded by a pair of the α_j, then all the zeros of $r'(z)$ interior to C lie on K, and lie on the two arcs of K bounded by C which are the shortest arcs of K interior to C terminated at one end by C and containing respectively all the points α_j and all the points β_k.

The function $r(z)$ of Theorem 5 is the quotient of two arbitrary functions of the kind defined by (1).

We shall establish Theorem 5 by means of the easily proved lemma:

LEMMA 2. *Let L be a circle orthogonal to C: $|z| = 1$, and let a particle of mass $+1$ be situated at α interior to C but not on L, and a particle of mass -1 be situated at $1/\bar{\alpha}$. Then the corresponding force at a point P of L interior to C has a nonvanishing component orthogonal to L in the sense directed from the side of C on which α lies.*

Lemma 2 may be proved at once by inversion in the unit circle whose center is P. Under this inversion let C, L, α, $1/\bar{\alpha}$ correspond,

respectively, to C', L', α', β', so that L' is a line passing through the center of the (proper) circle C'; the points α' and β' are mutually inverse with respect to C' and are both separated or both not separated from P by L', unless L is a straight line and consequently P lies on L'. But α' lies exterior to C' and β' interior to C'. The force at P due to the two given particles is represented by the sum of the vectors $\alpha'P$ and $P\beta'$; that is to say, is represented by the vector $\alpha'\beta'$. The vector $\alpha'\beta'$ has a nonvanishing component orthogonal to L', in the sense of the perpendicular from α' onto L'. This is equivalent to the statement of the lemma.

The proof of Theorem 5 is now immediate. Let L be orthogonal to C and separate each α_j from each β_k. Consider the field of force whose positions of equilibrium determine the zeros of $r'(z)$. Then the force at a point P of L interior to C due to the particles α_j and $1/\bar{\alpha}_j$ has a nonvanishing component in the sense directed from the side of L on which α_j lies. The force at P due to the particles β_k and $1/\bar{\beta}_k$ has a nonvanishing component in this same sense, so P cannot be a position of equilibrium. But P cannot be a multiple zero of $r(z)$, and hence cannot be a zero of $r'(z)$. Theorem 5 is established.

Theorem 4 can be proved from Theorem 5 by a limiting process.

Theorem 5 essentially includes Theorem 2, where the points β_k of Theorem 5 do not exist. Theorem 2 may also be established from Lemma 2. Indeed, both lemmas as well as Theorems 2 and 5 can be proved from the fact that in the field of force due to two particles of masses equal in magnitude but opposite in sign, the lines of force are precisely the circles through those particles. The sense of the force at every point of such a circle is directed from the repelling particle toward the attracting particle.

It follows that in the last part of Theorem 5 an arc of K interior to C bounded by two α_k (or by two β_k) and containing in its interior no α_k or β_k contains in its interior at least (as a matter of fact, precisely) one zero of $r'(z)$. A similar remark applies to the Corollary to Theorem 2, to the latter part of Theorem 6, and under suitable conditions to Theorem 8.

Under the conditions of the last part of Theorem 5, it may occur that zeros of $r'(z)$ lie on C but not on K, as we now show by means of an example. Let K be an arbitrary proper circle orthogonal to C, and let M denote the diameter of K through $O: z=0$. Let α_1 be chosen on K interior to C but not on M, let β_1 be the inverse of α_1 in M, which is also on K, and choose $m=n=1$. Considerations of symmetry and the fact that the lines of force due to the particles at α_1 and $1/\bar{\alpha}_1$ are circles through those points, show that the two intersections of M

with C are zeros of $r'(z)$; but these intersections do not lie on the circle K.

4. Derivative of a Blaschke product. The Blaschke product (assumed convergent) defined by (2) is the limit for $|z|<1$ of the sequence

$$B_n(z) = \prod_{k=1}^{k=n} \frac{\bar{\alpha}_k}{|\alpha_k|} \frac{z-\alpha_k}{\bar{\alpha}_k z - 1},$$

and $B_n(z)$ is of type (1). Convergence of $B_n(z)$ to $B(z)$ is uniform for $|z| \leqq r<1$, and convergence of the sequence $B_n'(z)$ to the function $B'(z)$ is also uniform in the closed region $|z| \leqq r<1$. If a point z_0 with $|z_0|<1$ is not a limit point of zeros of the functions $B_n'(z)$, it follows by a well known theorem due to Hurwitz that z_0 cannot be a zero of $B'(z)$. Consequently, we have from Theorem 2 the following theorem:*

THEOREM 6. *If a circular region G bounded by a circle Γ orthogonal to C: $|z|=1$ contains in its interior no point α_k, then the region G contains in its interior no zero of $B'(z)$ in $|z|<1$, where $B(z)$ is defined by* (2).

In particular, if all the points α_k lie on Γ, so also do all the zeros of $B'(z)$ in $|z|<1$.

A somewhat simpler but less precise result than Theorem 6 is the next theorem:

THEOREM 7. *Let the points α_k lie interior to a circle C_1 which is tangent to C: $|z|=1$ internally. Then all the finite zeros of $B'(z)$ interior to C lie in C_1.*

For each n, all of the points $\alpha_1, \alpha_2, \cdots, \alpha_n$ lie in a circle interior to but concentric with C_1; so by Theorem 1 the zeros of the derivatives of the corresponding partial product interior to C lie in the interior of this smaller circle. Theorem 7 thus follows from Hurwitz's theorem.

A similar method of proof yields an analogue of Theorem 5:

THEOREM 8. *Let $B(z)$ be the Blaschke product* (2), *and let*

$$D(z) = \prod_{k=1}^{\infty} \frac{\bar{\beta}_k}{|\beta_k|} \frac{z-\beta_k}{\bar{\beta}_k z - 1}, \qquad |\beta_k| < 1,$$

* It is to be remarked in connection with Theorem 6 that an arbitrary function $f(z)$ analytic and bounded for $|z|<1$, which has boundary values of constant modulus for normal approach to $|z|=1$, is a constant multiple of a Blaschke product of type (2).

also be a convergent Blaschke product. If L is a circle orthogonal to $C: |z| = 1$, and if L separates each α_j from each β_k, then no zero of $r'(z)$, where $r(z) \equiv B(z)/D(z)$, lies on L interior to C. Consequently, if such a circle L exists, all the zeros of $r'(z)$ interior to C lie in two closed point sets Π_1 and Π_2 which are bounded by circles orthogonal to C each of which separates all the α_j not lying on it from all the β_k not lying on it; these two point sets Π_1 and Π_2 are separated by every L; they contain respectively all the α_j and all the β_k.

In Theorem 3 we have emphasized the location of the zeros of $r'(z)$ *interior to* C. This is no actual restriction, for each result applies effectively also to the zeros of $r'(z)$ even exterior to C. For instance if we set $r_1(z) \equiv 1/r(1/z)$, where $r(z)$ is represented by (1), we have

$$r_1(z) \equiv \frac{1}{\lambda} \prod_{k=1}^{m} \frac{z - \bar{\alpha}_k}{1 - \alpha_k z}, \qquad |\bar{\alpha}_k| < 1,$$

so that Theorem 3, for example, applies to the function $r_1(z)$, and yields results on the zeros of $r'(z)$ exterior to C. In fact, all finite zeros of $r'(z)$ exterior to C lie in the inverse with respect to C of the polygon Π of Theorem 3. But if no α_k vanishes, the derivative $r'(z)$ vanishes at infinity whether or not the point at infinity lies in the inverse in C of the polygon Π.

A remark similar to the one just made applies not only to Theorem 3 but also to Theorems 5, 6, 7, and 8. But the present results do not treat directly the zeros of $r'(z)$ on C in Theorems 5 and 8.

A linear transformation of the complex variable which transforms $C: |z| = 1$ into the axis of reals yields for Theorems 1–3, 5–8 analogous results on the derivatives of functions of the forms

$$\prod_{k=1}^{m} \frac{z - \alpha_k}{z - \bar{\alpha}_k}, \qquad \prod_{k=1}^{\infty} \frac{z - \alpha_k}{z - \bar{\alpha}_k}.$$

5. **Critical points of harmonic functions.** By methods that the present writer has developed elsewhere,* the new results just established can be extended to apply to the critical points of harmonic functions. The detailed proof, which we leave to the reader, may be based directly on the theorems already proved.

THEOREM 9. *In the extended plane let C be a circle, let the Jordan curves $C_1', C_2', \cdots, C_r'$ not intersect C, all lie on the same side of C, and be respectively symmetric to the Jordan curves $C_1'', C_2'', \cdots, C_r''$ with*

* Proceedings of the National Academy of Sciences, vol. 20 (1934), pp. 551–554.

respect to C. Let no two of the curves C_j', and no two of the curves C_k'' intersect, and let the totality of those curves constitute the complete boundary of a region R. Let the function $u(x, y)$ be harmonic in the interior of R and continuous in the corresponding closed region, and let $u(x, y)$ take on the value unity on every C_k' and the value minus unity on every C_k''.

(1) If G is a region bounded by a circle Γ orthogonal to C, and if G contains in its interior no point exterior to R, then G contains in its interior no critical point of $u(x, y)$.

(2) If a circle K' separates all the curves C_k' from C, then K' separates precisely $\nu-1$ critical points of $u(x, y)$ from C; the inverse K'' of K' in C separates precisely $\nu-1$ critical points of $u(x, y)$ from C.

THEOREM 10. *In the extended plane let C be a circle, let the Jordan curves C_1', C_2', $\cdots$, C_μ', K_1', K_2', $\cdots$, K_ν' not intersect C, all lie on the same side of C, and be respectively symmetric to the Jordan curves C_1'', C_2'', $\cdots$, C_μ'', K_1'', K_2'', $\cdots$, K_ν'' with respect to C. Let no two of the curves C_1', $\cdots$, C_μ', K_1', $\cdots$, K_ν' intersect, and let the totality of the curves C_j', K_j', C_j'', K_j'' constitute the complete boundary of a region R. Let the function $u(x, y)$ be harmonic in the interior of R and continuous in the corresponding closed region, and let $u(x, y)$ take on the value unity on every C_j' and K_j'' and the value minus unity on every C_j'' and K_j'.*

If L is a circle orthogonal to C, and if L separates each C_j' from each K_k', then no critical point of $u(x, y)$ lies on L but not on C. Consequently, if such a circle L exists, the critical points of $u(x, y)$ not on C lie in two regions Π_1 and Π_2 which are bounded by circles L' orthogonal to C each of which separates (except for points of intersection with L') each C_j' from each K_k', and each C_j'' from each K_k''. The regions Π_1 and Π_2 are separated by every L; they contain, respectively, all the C_j' and C_j'', and all the K_j' and K_j''.

In Theorem 10 we may make the convention that the case $\nu=0$ is not excluded, and that the corresponding restriction on L is that L shall intersect no C_j' and shall separate no two of the circles C_j'. If this convention is made, Theorem 10 includes the essence of Theorems 1, 2, 3, 4, 5, 9, and by a limiting process may be used to prove also Theorems 6, 7, and 8.

Theorems 9 and 10 extend to certain situations in which the boundary of the region R has a finite or infinite number of components which are not necessarily Jordan curves.

HARVARD UNIVERSITY

A NOTE ON THE WEIERSTRASS CONDITION IN THE CALCULUS OF VARIATIONS*

M. R. HESTENES AND W. T. REID

The present note is concerned with the proof of a simple property of the Weierstrass E-function which, so far as the authors know, has not been pointed out before. For the sake of generality, the result will be given for the general problem of the Lagrange or Bolza type. The result for such a problem in non-parametric form is given in Theorem I below, and the analogous result for the parametric problem is presented in Theorem II.

For the non-parametric problem let $f(x, y_1, \cdots, y_n, p_1, \cdots, p_n) = f(x, y, p)$ denote the integrand function and $\phi_\alpha(x, y, p)$, $(\alpha = 1, \cdots, m < n)$, the auxiliary expressions.† It will be assumed that the functions f, ϕ_α are continuous and have continuous derivatives of the first two orders in a region‡ $\mathcal{R}$ of (x, y, p)-space. By an *admissible set*

$$(x, y_1, \cdots, y_n, p_1, \cdots, p_n, \lambda_1, \cdots, \lambda_m) = (x, y, p, \lambda)$$

will be meant one such that (x, y, p) is in $\mathcal{R}$ and satisfies the equations $\phi_\alpha = 0$. Let $F(x, y, p, \lambda) = f(x, y, p) + \lambda_\alpha \phi_\alpha(x, y, p)$. Here and elsewhere in this note the tensor analysis summation convention is used.

THEOREM I. *Suppose N is a region in (x, y, p, λ)-space such that at each admissible set (x, y, p, λ) of N the inequality*

$$E(x, y, p, \lambda, q) \equiv F(x, y, q, \lambda) - F(x, y, p, \lambda) - (q_i - p_i) F_{p_i}(x, y, p, \lambda) \geqq 0 \tag{1}$$

holds for every set (q_i) for which (x, y, q, λ) is admissible. If the matrix

$$\begin{Vmatrix} F_{p_i p_j} & \phi_{\beta p_i} \\ \phi_{\alpha p_j} & 0 \cdot \delta_{\alpha\beta} \end{Vmatrix}, \qquad i, j = 1, \cdots, n; \alpha, \beta = 1, \cdots, m, \tag{2}$$

* Presented to the Society, December 30, 1938.

† For a more detailed formulation of the problems of Lagrange and Bolza the reader is referred, for example, to Bliss, *The problem of Lagrange in the calculus of variations*, American Journal of Mathematics, vol. 52 (1930), pp. 673–742, Morse, *Sufficient conditions in the problem of Lagrange with variable end-conditions*, American Journal of Mathematics, vol. 53 (1931), pp. 517–546, or Bliss, *The Problem of Bolza in the Calculus of Variations*, mimeographed notes of lectures delivered winter, 1935, at the University of Chicago.

‡ By "region" we shall understand "open region." It is to be noted that in the following theorems no use is made of the region's being open with respect to the (x, y) or (y) variables. Consequently, the hypotheses of the theorems could be weakened in this respect.

is nonsingular at each admissible set (x, y, p, λ) *in* N, *the equality in* (1) *holds only in case* $p_i = q_i$, $(i = 1, \cdots, n)$.

For suppose that the equality in (1) holds for a particular admissible set $(x_0, y_0, p_0, \lambda_0)$ of N and a set q_0 with $(x_0, y_0, q_0, \lambda_0)$ admissible. It would then follow that (p_0, λ_0) affords $E(x_0, y_0, p, \lambda, q_0)$ a minimum relative to neighboring sets (p, λ) for which $\phi_\alpha(x_0, y_0, p) = 0$, $(\alpha = 1, \cdots, m)$. Consequently, by the Lagrange multiplier rule,* there would exist multipliers l_α such that at $(x_0, y_0, p_0, \lambda_0, q_0)$ we have

$$E_{p_j} + l_\alpha \phi_{\alpha p_j} = 0, \qquad E_{\lambda_\alpha} = 0, \qquad j = 1, \cdots, n; \alpha = 1, \cdots, m;$$

that is,

$$(3) \qquad -(q_{i0} - p_{i0})F_{p_i p_j} + l_\alpha \phi_{\alpha p_j} = 0, \quad -(q_{i0} - p_{i0})\phi_{\alpha p_i} = 0.$$

But in view of the nonsingularity of the matrix (2) at admissible sets in N, these equations imply $q_{i0} = p_{i0}$, $l_\alpha = 0$; hence the theorem is established.

For the problem of the calculus of variations in parametric form, the functions f, ϕ_α, $(\alpha = 1, \cdots, m < n-1)$, are assumed to be independent of the variable x and to be positively homogeneous of degree one in the variables p_i. It is also assumed that these functions are continuous and have continuous derivatives of the first two orders in a region $\mathcal{R}$ of (y, p)-space which is such that if (y, p) is in $\mathcal{R}$ then $p_i p_i \neq 0$, and, moreover, the point (y, kp), $(k > 0)$, is also in $\mathcal{R}$. An admissible set (y, p, λ) is defined in a manner analogous to that used above. One has the well known relations

$$(4) \qquad F(y, p, \lambda) = p_i F_{p_i}(y, p, \lambda), \qquad \phi_\alpha(y, p) = p_i \phi_{\alpha p_i}(y, p), \qquad p_i F_{p_i p_j}(y, p, \lambda) = 0,$$

$$(5) \qquad F_{p_i}(y, kp, \lambda) = F_{p_i}(y, p, \lambda), \qquad k > 0; i, j = 1, \cdots, n; \alpha = 1, \cdots, m.$$

From the second and third relations of (4) it follows that at an admissible set (y, p, λ) the matrix (2) is singular. It is also a consequence of (5) and the first equation of (4) that

$$(6) \qquad E(y, p, \lambda, q) = q_i[F_{p_i}(y, q, \lambda) - F_{p_i}(y, p, \lambda)],$$

$$(7) \qquad E(y, kp, \lambda, k'q) = k' E(y, p, \lambda, q), \qquad k > 0, k' > 0.$$

THEOREM II. *Suppose* N *is a region in* (y, p, λ)*-space such that at each admissible set* (y, p, λ) *of* N *the inequality*

* See, for example, Carathéodory, *Variationsrechnung*, p. 116.

$$E(y, p, \lambda, q) \geqq 0 \tag{8}$$

holds at every set (q_i) *for which* (y, q, λ) *is admissible. If the matrix* (2) *is of rank* $n+m-1$ *at each admissible set* (y, p, λ) *in* N, *the equality in* (8) *holds only in case* $q_i = hp_i$, $(h>0)$.

Suppose the equality in (8) holds for a particular admissible set (y_0, p_0, λ_0) of N and a set q_0 with (y_0, q_0, λ_0) admissible. In view of (7) we may assume that $p_{i0}p_{i0}=1$, $q_{i0}q_{i0}=1$. Proceeding as in the proof of Theorem I, we find that at the set $(y_0, p_0, \lambda_0, q_0)$ equations (3) hold. By virtue of relations (4) and the fact that the matrix (2) is of rank $n+m-1$ at (y_0, p_0, λ_0), equations (3) imply $q_{i0}=hp_{i0}$, $l_\alpha=0$; moreover, since $p_{i0}p_{i0}=1$, $q_{i0}q_{i0}=1$, we have $h=\pm 1$. The inequality in (8) holds, therefore, for every set (y, p, λ, q) satisfying $p_ip_i=1$, $q_iq_i=1$, $(q_i)\neq\pm(p_i)$, (y, p, λ) an admissible set in N, (y, q, λ) an admissible set. But from the form (6) of $E(y, p, \lambda, q)$ one readily verifies that if $(y_0, -p_0)$ is in $\mathfrak{R}$, then

$$E(y_0, p_0, \lambda_0, -p_0) = E(y_0, r, \lambda_0, -p_0) + E(y_0, r, \lambda_0, p_0) \tag{9}$$

for every set r such that (y, r) is in $\mathfrak{R}$. In particular, since (y_0, p_0, λ_0) is an admissible set in N, it follows from the usual implicit function theorem that there exists a neighboring admissible set (y_0, r, λ_0) in N such that $r_ir_i=1$, $(r_i)\neq\pm(p_{i0})$. When this value of r is substituted in (9), it is found that $E(y_0, p_0, \lambda_0, -p_0)>0$ if $(y_0, -p_0, \lambda_0)$ is admissible. Consequently, whenever (y, p, λ) is an admissible set in N, (y, q, λ) is an admissible set, and $p_ip_i=1$, $q_iq_i=1$, the equality in (8) holds if and only if $p_i=q_i$. From relation (7) this result is readily seen to be equivalent to the conclusion of Theorem II.

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ON THE NUMBER OF SETS CONJUGATE TO A MATRIX WITH LINEAR ELEMENTARY DIVISORS*

H. S. THURSTON

1. **Introduction.** The determination of sets of conjugates to a given matrix M with respect to its minimum equation was first discussed by E. S. Sokolnikoff in 1933.† In two papers published more recently, one‡ by E. T. Browne and one§ by the writer, the subject has received further consideration, and, in particular, attention has been paid to the case in which M has only linear elementary divisors. In the former of these two papers, conjugates were obtained by employing the principal idempotent elements of M. The second paper considered only conjugates of M in a ring $R(A)$ where A had simple latent roots, and made use of the principal idempotent elements of A.

Since each principal idempotent element of M is a set of one or more of its partial idempotent elements, there will be, in general, more sets of conjugates obtained by using the latter than by using the former. This fact was stated by both Sokolnikoff and Browne. Not all of these conjugates are polynomials in M. The writer exhibited sets of matrices in the ring $R(A)$ conjugate to M but not expressible as polynomials in M. These are accounted for by the fact that the principal idempotent elements of A are the partial idempotent elements of M.

It is the purpose of this paper to show that there exists a nonderogatory matrix A (by no means unique), such that any matrix M can be expressed as a polynomial in A. Then for the restricted case under consideration, the exact numbers of conjugate sets in the rings $R(A)$ and $R(M)$ can be determined.

2. **Determination of the matrix A.** Let M be any square matrix of order n. Then there exists a nonsingular matrix T such that $T^{-1}MT=C$, where C is in the Jordan canonical form. Let B be any matrix in Jordan canonical form whose nonoverlapping blocks are of the same order as those in C, and having no one latent root occurring in two such blocks. It is easily shown that C is an integral, rational function of B, say $f(B)$. Then, if $A=TBT^{-1}$, it follows that

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* Presented to the Society, September 6, 1938.

† E. S. Sokolnikoff, American Journal of Mathematics, vol. 55 (1933), pp. 167–180.

‡ E. T. Browne, ibid, vol. 59 (1937), pp. 845–868.

§ H. S. Thurston, this Bulletin, vol. 44 (1938), pp. 258–261.

$$M = TCT^{-1} = T \cdot f(B) \cdot T^{-1} = f(TBT^{-1}) = f(A).$$

Hence we have the following theorem:

THEOREM 1. *If M is any square matrix, there can be found a non-derogatory matrix A such that M is expressible as a polynomial in A.*

3. **Sets of conjugates to M.** We shall now restrict our attention to the case where M has only linear elementary divisors. The matrix A determined in the above theorem then has only simple latent roots. If the characteristic function of M is $\prod_{i=1}^{r}(\lambda-\alpha_i)^{\nu_i}$, its minimum equation is $F(\lambda) \equiv \prod_{i=1}^{r}(\lambda-\alpha_i) = 0$.

Let e_i, $(i=1, 2, \cdots, n)$, and ϕ_j, $(j=1, 2, \cdots, r)$, be the principal idempotent elements of A and of M, respectively. Then $M = \sum_{i=1}^{n}\beta_i e_i$ where ν_1 of the β's are α_1, ν_2 are α_2, $\cdots$, and ν_r are α_r. We shall denote the matrix M by the symbol $(\beta_1, \beta_2, \cdots, \beta_n)_{e_i}$. Similarly, we have $M = \sum_{j=1}^{r}\alpha_j\phi_j$ expressed symbolically by $M = (\alpha_1, \alpha_2, \cdots, \alpha_r)_{\phi_j}$. Then every matrix ϕ_j is the sum of ν_j of the e_i, the latter being partial idempotent elements of M. For example, if $M = (1, 2, 1, 1, 3, 2)_{e_i} = (1, 2, 3)_{\phi_j}$, we have

$$\phi_1 = e_1 + e_3 + e_4, \qquad \phi_2 = e_2 + e_6, \qquad \phi_3 = e_5.$$

In another paper* the following theorem has been proved:

THEOREM 2. *If μ is the number of distinct latent roots of a matrix A, then any polynomial $F(\lambda)$, of degree t, with simple zeros, breaks up in $(t!)^{\mu-1}$ ways into a product of t linear factors in the ring $R(A)$.*

Since each factorization involves t matrices, all ways of factoring will involve $t \cdot (t!)^{\mu-1}$ matrices, and since $F(\lambda) = 0$ has t^{μ} matric roots equally distributed in the factorizations, it follows that, corresponding to any one of the t matric roots, there will be $t \cdot (t!)^{\mu-1}/t^{\mu}$ or $[(t-1)!]^{\mu-1}$ sets of conjugates. Applying this argument to $F(\lambda) \equiv \prod_{i=1}^{r}(\lambda-\alpha_i) = 0$, the minimum equation of M, we have the following theorem:

THEOREM 3. *The matrix $M = (\beta_1, \beta_2, \cdots, \beta_n)_{e_i} = (\alpha_1, \alpha_2, \cdots, \alpha_r)_{\phi_j}$ has, in the ring $R(A)$, $[(r-1)!]^{n-1}$ sets of conjugates with respect to its minimum equation; the matrices of $[(r-1)!]^{r-1}$ of these sets are integral, rational functions of M.*

EXAMPLE. If $M = (1, 2, 1, 3)_{e_i} = (1, 2, 3)_{\phi_j}$, we have the following sets of conjugates in $R(A)$:

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* Compositio Mathematica, vol. 6 (1938), pp. 235–238.

I	II
(1,2,1,3), (2,3,2,1), (3,1,3,2),	(1,2,1,3), (2,1,3,1), (3,3,2,2),
(1,2,1,3), (2,1,2,2), (3,3,3,1),	(1,2,1,3), (2,1,3,2), (3,3,2,1),
(1,2,1,3), (2,3,2,2), (3,1,3,1),	(1,2,1,3), (2,3,3,1), (3,1,2,2),
(1,2,1,3), (2,1,2,1), (3,3,3,2),	(1,2,1,3), (2,3,3,2), (3,1,2,1),

while in $R(M)$ we have

III

(1,2,3), (2,3,1), (3,1,2),
(1,2,3), (2,1,2), (3,3,1),
(1,2,3), (2,3,2), (3,1,1),
(1,2,3), (2,1,1), (3,3,2).

Since $\phi_1 = e_1 + e_3$, $\phi_2 = e_2$, $\phi_3 = e_4$, it is not difficult to identify the sets in group III with those in group I. Thus

$$(1, 2, 3)_{\phi_j} = \phi_1 + 2\phi_2 + 3\phi_3 = e_1 + 2e_2 + e_3 + 3e_4 = (1, 2, 1, 3)_{e_i},$$

and similarly $(2, 3, 2)_{\phi_j} = (2, 3, 2, 2)_{e_i}$, $(3, 1, 1)_{\phi_j} = (3, 1, 3, 1)_{e_i}$.

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TRIPLE SYSTEMS AS RULED QUADRICS*

W. G. WARNOCK

1. **Introduction.** If n elements $x_1, x_2, \cdots, x_n$ can be arranged in triples such that each pair $x_i x_j$ occurs in one and only one triple, the arrangement so formed is a simple triple system. Credit for the first published paper on such systems is given to Kirkman.† Methods of construction, properties, and forms of interpretation of these and more general multiple systems can be found throughout the mathematical literature since that date.‡ In this note I propose to treat the element as a generic line in an ordinary three-space. Likewise, I shall point out some of the group properties which seem worthy of men-

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† Kirkman, The Cambridge and Dublin Mathematical Journal, vol. 2 (1847), pp. 192–204.

‡ See A. Emch, *Triple and multiple systems, their geometric configurations and groups*, Transactions of this Society, vol. 31 (1929), pp. 25–42. An almost complete list of references is given in this paper. A more recent discussion of multiple systems is to be found in an article by R. D. Carmichael, *Tactical configurations of rank two*, American Journal of Mathematics, vol. 53 (1931), pp. 217–240.

tion. This note is restricted to simple triple systems on seven elements.

An example of a triple system on seven elements is

$$(1) \qquad \begin{matrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 2 & 4 & 5 & 4 & 5 & 4 & 6 \\ 3 & 7 & 6 & 6 & 7 & 5 & 7 \end{matrix}.$$

It is well known that this system is invariant under a group G_{168}, the simple group of order 168. There are $7!/168 = 30$ distinct triple systems on seven elements. Cayley* showed that any 6 distinct systems contained all the 35 possible couples but that 5 systems cannot be formed to include all these couples. The 30 distinct systems are listed at the end of this article and shall be referred to by the corresponding numbers to their left.

2. **Elements interpreted as lines.** When one considers an element as a skew line in an S_3, a triple represents three skew lines. These lines may be taken as three rulings on a ruled quadric surface formed by all the lines that intersect these three rulings. A simple triple system Δ_7^3, represents seven such quadric surfaces. We examine first the nature of the intersections of these surfaces. Upon examination of (1) in §1 it is seen that the ruled quadric 123 intersects the ruled quadrics 147 and 156 in the ruling 1; 123 intersects 246 and 257 in ruling 2; likewise, 123 intersects 345 and 367 in ruling 3. Similarly the other intersections may be counted. If two ruled quadrics intersect along a ruling, the remaining portion of the intersection is a space cubic. This residual cubic cuts the common ruling in two points, as is easily shown. There are 21 residual cubics in the total system of a simple Δ_7^3. Each cubic cuts 2 rulings, and each ruling intersects 15 of the 21 cubics. On each surface lie 6 cubics. If one projects upon any plane from a point on a common ruling, the residual cubic becomes a plane cubic with a double point. The residual cubics are then rational, and the 15 which intersect a given ruling project into a system of 15 plane cubics having a common double point.

3. **Triple systems with no common triple.** There exist among the 30 distinct Δ_7^3 systems certain ones which have no triple in common with any given Δ_7^3. On seven elements these do have common couples, however. The systems (1) and (19) are two such systems. One might ask for a triple system somewhat more general than a simple system and demand that the n elements form a triple system with every

* Cayley, *Collected Papers*, vol. 1, pp. 481–484.

couple taken twice. If x represents the number of triples in such a system, it develops that $x=n(n-1)/3$. Integral solutions are possible when n is of the form $3k$ or $3k+1$. In the case under consideration, $n=7$ and $x=14$; however, for our purposes, it is better to consider this not as a system of 14 triples on 7 elements but as two simple systems without a common triple but with common couples.*

The operators

$$S_1 = (2\ 7)(3\ 5)(4\ 6), \quad S_2 = (1\ 5)(3\ 4)(6\ 7), \quad S_3 = (1\ 6)(2\ 5)(4\ 7),$$
$$S_4 = (1\ 2)(3\ 6)(5\ 7), \quad S_5 = (1\ 4)(2\ 6)(3\ 7), \quad S_6 = (1\ 3)(4\ 5)(3\ 6),$$
$$S_7 = (1\ 3)(2\ 4)(5\ 6)$$

leave the elements 1, 2, 3, 4, 5, 6, 7 of the systems (1) and (19) invariant. The product $S_1S_2=(1\ 5\ 4\ 7\ 2\ 6\ 3)$ is a cyclic substitution; so all of its powers are in the group and likewise are products S_iS_j. The group which leaves the two systems invariant is then a $G_{14}=I$, $S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, T, T^2, T^3, T^4, T^5, T^6$.

In terms of ruled quadrics, it is seen that each quadric intersects 9 other quadrics, 6 of its own system and 3 of its complementary system, in a common ruling and a residual cubic; 63 cubics are so formed. Again, a triple of one system has a couple in common with three triples of the complementary system. Geometrically these quadrics will have two rulings in common, leaving conics for the residual curves of intersection. This gives $7\cdot 3=21$ conics of intersection in two systems of this sort.

To find the number of distinct triple systems with no triple in common with (1), notice that the operator S_1 leaves the element 1 invariant and permutes the remaining elements in such a manner that no couple remains unchanged. If this were not demanded, a triple would remain unchanged, and thereby a Δ_7^2 would be created with at least one triple in common with (1). With this restriction there are only 8 ways in which the element 1 can remain invariant. To these 8 permutations correspond distinct triple systems. Each system is invariant with (1) under a group of order 14 as is indicated above. These systems having common couples but no common triples with (1) are (9), (12), (15), (16), (19), (23), (26), (29) and may be obtained by operating upon (1), respectively, by

$$O_1 = (2\ 6)(3\ 7)(4\ 5), \quad P_1 = (2\ 5)(3\ 7)(4\ 6), \quad Q_1 = (2\ 7)(3\ 6)(4\ 5),$$
$$R_1 = (2\ 4)(3\ 6)(5\ 7), \quad S_1 = (2\ 7)(3\ 5)(4\ 6), \quad U_1 = (2\ 4)(3\ 5)(6\ 7),$$
$$V_1 = (2\ 5)(3\ 4)(6\ 7), \quad W_1 = (2\ 6)(3\ 4)(6\ 7).$$

* Emch, loc. cit., p. 29.

There is no group under which these 9 systems are invariant; for the product of two operators above leaves a triple invariant and hence cannot belong to the system.

4. **Systems with one triple in common.** Some of the 30 systems have 1 triple in common with (1). The system (2) furnishes an example with the triple 123 in common. The corresponding common ruled quadric has 12 residual cubics of intersections, 6 from each system, on its surface. On any other ruled quadric of the two systems lie 8 residual cubics, 6 by intersections with surfaces of its own system and 2 by intersections with surfaces of its complementary system. Sixty cubics are so formed by two systems of this type. The total number of conics formed by common couples of lines as part of the intersection is 21.

The group that leaves two such systems invariant must leave the common triple invariant. It can be expressed in terms of the remaining four elements not in this common triple. This group must be the symmetric group G_{24} or one of its subgroups. It cannot contain a transposition; for, in that case, at least one other triple would remain invariant. As this is the only restriction necessary, the desired group is the alternating group G_{12} on the elements not contained in the common triple.

There are 14 of the 30 distinct Δ_7^3 systems which have one triple in common with (1). They are (2), (3), (7), (8), (10), (13), (17), (18), (20), (21), (24), (25), (28), (30). If a couple is considered fixed, a triple can be formed in five ways by taking one of the remaining elements. This fixes the common triple and likewise fixes one element in each of the remaining triples if the process is considered as a permutation upon (1). The remaining places in the second and third triples (columns of (1)) are to be chosen from the remaining four elements, and the filling of these spaces determines the respective positions of these four elements in the last four triples. This can be done in $4 \cdot 3/2 = 6$ ways, but the arrangement of columns in a system is nonessential so that finally there are $5 \cdot 6/2 = 15$ possible arrangements. These include (1); the other Δ_7^3 systems have been listed above.

5. **Systems with three triples in common.** The 7 remaining Δ_7^3, systems have three triples in common with (1). Each of these is invariant with (1) under a group of order 2^3 and type (1, 1, 1).

From the fact that there are $7 \cdot 6/2 = 21$ ways of forming a transposition substitution and the fact that the invariant group is of the type (1, 1, 1), it follows that there are $21/3 = 7$ distinct Δ_7^3 systems of this nature. These are (4), (5), (6), (11), (14), (22), (27).

In terms of the ruled quadrics and their intersections, there lie 10 residual cubics on each of the common quadrics of the two complementary systems and 6 on each of the other quadric surfaces. On each of these surfaces lie 3 conics. This gives 36 cubics and 12 conics as curves of intersection of (1) with each of the seven systems above. This completes the count of the simple systems.

LIST OF DISTINCT Δ_7^3 SYSTEMS

	1	1	1	2	2	3	3		1	1	1	2	2	3	4		1	1	1	2	2	3	4
(1)	2	5	4	5	4	4	6	(11)	2	3	5	3	5	4	6	(21)	2	3	4	5	3	5	6
	3	6	7	7	6	5	7		4	7	6	6	7	5	7		6	7	5	7	4	6	7
	1	1	1	2	2	3	3		1	1	1	2	2	3	4		1	1	1	2	2	3	4
(2)	2	4	5	4	6	5	4	(12)	2	3	5	3	6	4	5	(22)	2	3	4	3	5	6	5
	3	6	7	5	7	6	7		4	6	7	5	7	7	6		6	5	7	4	7	7	6
	1	1	1	2	2	3	3		1	1	1	2	2	3	4		1	1	1	2	2	3	5
(3)	2	4	6	4	5	5	4	(13)	2	3	6	3	4	5	5	(23)	2	3	4	4	3	4	6
	3	5	7	7	6	7	6		5	4	7	7	6	6	7		6	7	5	7	5	6	7
	1	1	1	2	2	3	3		1	1	1	2	2	3	5		1	1	1	2	2	3	4
(4)	2	4	3	4	6	4	5	(14)	2	3	4	4	3	4	6	(24)	2	3	5	3	4	6	5
	3	7	6	5	7	6	7		5	6	7	6	7	5	7		6	4	7	5	7	7	6
	1	1	1	2	2	3	3		1	1	1	2	2	3	4		1	1	1	2	2	3	5
(5)	2	4	6	4	5	4	5	(15)	2	3	6	4	3	5	5	(25)	2	3	4	5	3	6	5
	3	5	7	6	7	7	6		5	4	7	7	6	7	6		7	5	6	6	4	7	7
	1	1	1	2	2	3	3		1	1	1	2	2	3	4		1	1	1	2	2	3	4
(6)	2	4	5	5	4	4	6	(16)	2	3	4	3	6	5	5	(26)	2	4	3	3	5	5	6
	3	6	7	6	7	5	7		5	7	6	4	7	6	7		7	5	6	4	6	7	7
	1	1	1	2	2	3	4		1	1	1	2	2	3	4		1	1	1	2	2	6	7
(7)	2	3	5	5	3	4	6	(17)	2	3	4	3	6	5	5	(27)	2	3	5	3	4	6	5
	4	6	7	6	7	5	7		5	6	7	4	7	7	6		7	4	6	5	6	7	7
	1	1	1	2	2	3	4		1	1	1	2	2	3	5		1	1	1	2	2	3	5
(8)	2	3	6	3	5	4	5	(18)	2	3	4	3	4	4	6	(28)	2	3	4	3	4	4	6
	4	5	7	6	7	7	6		5	7	6	6	7	5	7		7	6	5	5	6	7	7
	1	1	1	2	2	3	4		1	1	1	2	2	3	4		1	1	1	2	2	3	5
(9)	2	3	6	5	3	4	5	(19)	2	3	5	3	4	5	6	(29)	2	3	4	3	4	4	6
	4	5	7	6	7	6	7		6	4	7	7	5	6	7		7	5	6	6	5	7	7
	1	1	1	2	2	3	4		1	1	1	2	2	3	5		1	1	1	2	2	3	4
(10)	2	3	6	3	7	4	5	(20)	2	3	4	3	4	4	6	(30)	2	3	5	3	4	5	6
	4	7	5	5	6	6	7		6	5	7	7	5	6	7		7	4	6	6	5	7	7

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